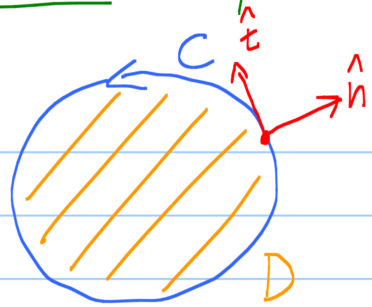
Lesson 16 Importance of Green's theorem



$$\int_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Divergence theorem: (Gauss' thm)

$$\int_C \vec{F}\cdot\hat{n}\,ds = \iint_D \underbrace{\mathrm{Div}\,\vec{F}}_{\frac{\partial F_1}{\partial x}+\frac{\partial F_2}{\partial y}}\,dx\,dy$$

$$\vec{F} = F_1\hat{\imath} + F_2\hat{\jmath}$$

Aha! Take $\begin{cases} F_1 = Q \\ F_2 = -P \end{cases}$

$$= \underbrace{\int_C (-F_2)\,dx + (F_1)\,dy}_{\text{This is the "flux"}}$$

Why: $C: (x(t), y(t)), 0 \le t \le L$

where $t =$ arc length

$$\hat{t} = \underbrace{\dot{x}(t)\hat{\imath} + \dot{y}(t)\hat{\jmath}}_{\text{unit vector}}$$

$$= \int_0^L F_2(-\dot{x}) + F_1(\dot{y})\,dt$$

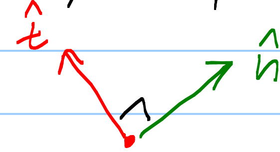
$$= \int_0^L (F_1, F_2)\cdot\underbrace{(\dot{y}, -\dot{x})}_{\text{Claim: } = \hat{n}}\,dt$$

Clear: $(\dot{x}, \dot{y})$ and $(\dot{y}, -\dot{x})$ are both unit vectors and $\perp$

Claim: $(\dot{y}, -\dot{x})$ points out.

Why: Represent vectors by complex #'s of same modulus.

$$\hat{t} \sim \dot{x} + i\dot{y}$$



$$\hat{n} \sim \underbrace{e^{-i\pi/2}}_{=-i}\,\hat{t} = -i\dot{x} + \dot{y}$$
$$= \dot{y} - i\dot{x}$$

$e^{i\alpha}(re^{i\theta})$
$= re^{i(\theta+\alpha)}$
rotates c-clockwise
by α

$$\int_C \underbrace{(F_1,F_2)}_{\vec{F}}\cdot\underbrace{(\dot{y},-\dot{x})}_{\hat{n}}\,\underset{\substack{\uparrow\\ \text{arc length}}}{dt} = \int_C \vec{F}\cdot\hat{n}\,ds$$

Green's 1st identity: $\displaystyle\int_C u\frac{\partial v}{\partial n}\,ds = \iint_D \nabla u\cdot\nabla v + u\Delta v\;dx\,dy$

Pf: Let $\vec{F} = u\nabla v$ in Div thm.

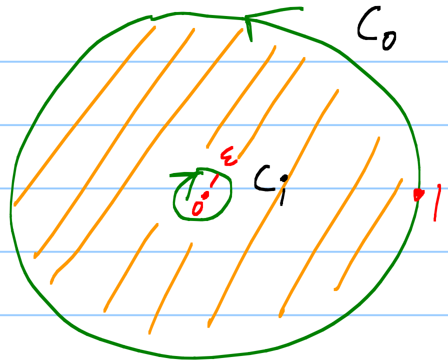
$$\int_C \vec{F}\cdot\hat{n}\,ds = \int_C u\underbrace{(\nabla v\cdot\hat{n})}_{\frac{\partial v}{\partial n}}\,ds = \iint_D \underbrace{\operatorname{Div}\vec{F}}_{\operatorname{Div}(u\nabla v)}\;dx\,dy$$

Green's 2nd identity: $\displaystyle\int_C \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right)ds = \iint_D u\Delta v - v\Delta u\;dx\,dy$

why:
$$\int_C u\frac{\partial v}{\partial n}\,ds = \iint_D (\underbrace{\nabla u\cdot\nabla v}_{\shortparallel} + u\Delta v)\,dx\,dy$$
$$-\quad \int_C v\frac{\partial u}{\partial n}\,ds = \iint_D (\overbrace{\nabla v\cdot\nabla u} + v\Delta u)\,dx\,dy$$

Can use Green's 2^{nd} to prove ave property for harmonic fcns (C^2-smooth satisfying the Laplace eqn $\Delta u \equiv 0$).

Pf: u harmonic, $v = \ln r$ ← harmonic!



$C = C_o \cup C_i$

$$\int_C \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) ds = \iint_D \left(u\underbrace{\Delta v}_{0} - v\underbrace{\Delta u}_{\overset{\shortparallel}{0}}\right) dx\,dy$$

$$\underbrace{\int_{C_o} u\underbrace{\frac{\partial v}{\partial n}}_{\frac{1}{r}=1 \text{ on } C_o} - \underbrace{v}_{v=0 \text{ on } C_o}\frac{\partial u}{\partial n}\,ds}_{\int_{C_o} u\,ds} - \underbrace{\int_{-C_i} u\underbrace{\frac{\partial v}{\partial n}}_{=\frac{1}{\varepsilon}} - v\frac{\partial u}{\partial n}\,ds}_{-\int_{-C_i} u\frac{1}{\varepsilon}\,ds \;+\; \int_{-C_i}\ln\varepsilon\,(\text{Bounded})\,ds} = 0$$

$\int_{-C_i}\ln\varepsilon\,(\text{Bounded})\,ds \to 0$ as $\varepsilon \to 0$

$$-\int_0^{2\pi} u(\varepsilon e^{it})\frac{1}{\varepsilon}(\varepsilon\,dt)$$

$\to -2\pi\, u(0)$ as $\varepsilon \to 0$

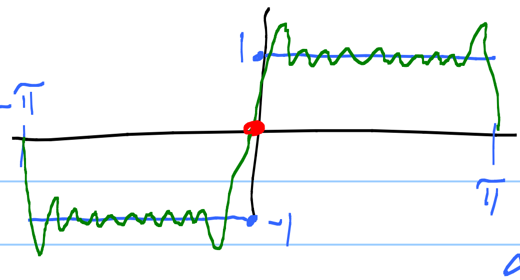
because u is continuous!

Done. Let $\varepsilon \to 0$ and $\int_{-C_i}$ on other side. ✓

Why do Fourier series converge to midpoint of jumps?

Just need to study the quintessential jump fcn!

$J(x)$:



$f(x)$ odd. ave $= 0$

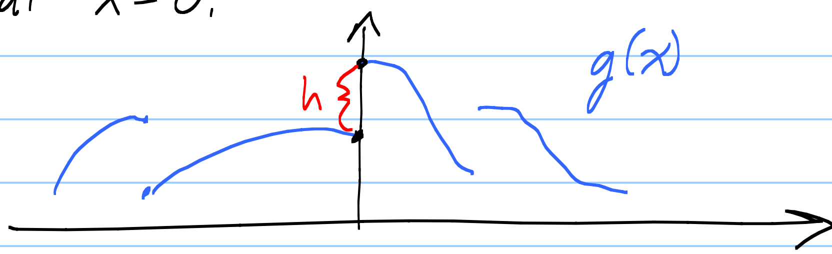
No cosine terms.

$a_0 = 0$

Get Fourier series: $0 + \sum_{n=1}^{\infty} b_n \sin nx$

Plug in $x=0$: Fourier series $= 0$.

__Idea:__ Given a piecewise C^1-smooth $g(x)$ with a jump at $x=0$.



Aha! $g(x) - \frac{h}{2} J(x)$ has no jump at $x=0$

It might have a kink at $x=0$, but no jump.

So F.S. for $g - \frac{h}{2} J$ converges to fcn there.

$$(\text{FS for } g) - \frac{h}{2}(\text{FS for } J)$$

Conclude that FS for g converges to midpoint of jump at $x=0$.

How to extend idea to $x \neq 0$:

$$T_a f(x) = f(x-a)$$

Guess: Fourier series of $T_a f$ is T_a of F. series.

True! Easy proof. [need f 2π-periodic]