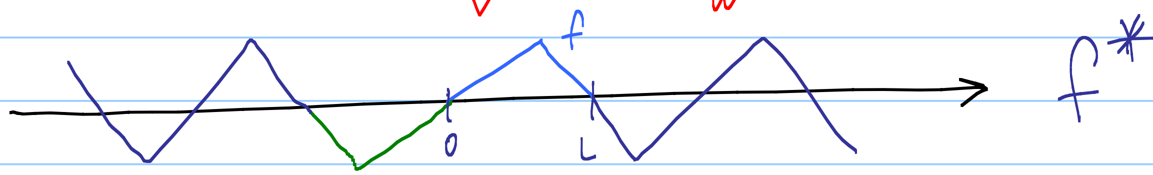
Lecture 21 Frosting on the cake: The isoperimetric inequality and Weyl's equidistribution theorem

Read Chapter 4

d'Alembert's solution to the string problem p.8 (1.2) Stein

Jacob Bernoulli's: $\frac{1}{2}\left[f^*(\underbrace{x+ct}_{v}) + f^*(\underbrace{x-ct}_{w})\right]$ ξ, η books



d'Alembert: $v = x+ct$
$w = x-ct$

$$u_x = \frac{\partial u}{\partial v}\underbrace{\frac{\partial v}{\partial x}}_{1} + \frac{\partial u}{\partial w}\underbrace{\frac{\partial w}{\partial x}}_{1} = u_v + u_w$$

$$u_{xx} = \left(\frac{\partial u_v}{\partial v}\underbrace{\frac{\partial v}{\partial x}}_{1} + \frac{\partial u_v}{\partial w}\underbrace{\frac{\partial w}{\partial x}}_{1}\right) + \left(\frac{\partial u_w}{\partial v}\underbrace{\frac{\partial v}{\partial x}}_{1} + \frac{\partial u_w}{\partial w}\underbrace{\frac{\partial w}{\partial x}}_{1}\right)$$

$$= u_{vv} + \underbrace{u_{wv} + u_{vw}}_{= \text{ if } C^2\text{-smooth}} + u_{ww}$$

$$= u_{vv} + 2u_{wv} + u_{ww}$$

Repeat in t var: $u_t = \frac{\partial u}{\partial v}\underbrace{\frac{\partial v}{\partial t}}_{c} + \frac{\partial u}{\partial w}\underbrace{\frac{\partial w}{\partial t}}_{-c}$

$$\vdots$$

$$u_{tt} = c^2\left(u_{vv} - 2u_{wv} + u_{ww}\right)$$

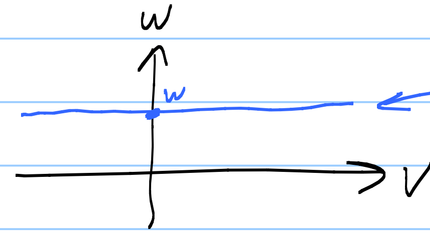
Plug into wave eqn: $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

Wow! Get $-2c^2 u_{wv} = 2c^2 u_{wv}$

New PDE : $\boxed{u_{wv} = 0}$ easy!

Freshman calculus : $\frac{d}{dx} h(x) \equiv 0 \implies h \equiv \text{const.}$ (MVT)

$*$ $\boxed{\frac{\partial u}{\partial v} \equiv 0}$:



← u const on horiz lines

Conclude $u(w,v) = C(w)$ ← "constant that might depend on w"

Solution to $(*)$: $u(w,v) =$ arbitrary fcn of w.

Our problem : $\frac{\partial}{\partial w}\left[\frac{\partial u}{\partial v}\right] \equiv 0$

Step 1 : $\frac{\partial u}{\partial v} =$ arb fcn of $v = C(v)$

$$\boxed{\frac{\partial u}{\partial v} = C(v)} \quad (+)$$

Step 2 : Let $p(v)$ be an antiderivative of $C(v)$. $p'(v) = C(v)$

$(+)$ $$\frac{\partial u}{\partial v} - p'(v) \equiv 0$$

$$\frac{\partial}{\partial v}\left[u - p\right] \equiv 0$$

So $u(w,v) - p(v) =$ arb fcn of w

$$= K(w)$$

Aha!

$$u(w,v) = p(v) + K(w) \quad !$$

D'Alembert's :
$$u(x,t) = \varphi(x+ct) + \psi(x-ct)$$

$$\frac{\partial u}{\partial t} = c\varphi'(x+ct) - c\psi'(x-ct)$$

I.C. :
$$\begin{cases} u(x,0) = \varphi(x) + \psi(x) \overset{\text{want}}{=} f(x) & (A) \\ \frac{\partial u}{\partial t}(x,0) = c\varphi'(x) - c\psi'(x) = g(x) & (B) \end{cases}$$

Fun case : harpsichord : $g(x) \equiv 0$

$$\begin{cases} (A): \quad \varphi(x) + \psi(x) = f(x) \\ (B): \quad \varphi'(x) - \psi'(x) = 0 \end{cases}$$

Get $\varphi = \frac{1}{2}f$, $\psi = \frac{1}{2}f$.

Last thing to wory about : BC
$$u(0,t) = 0$$
$$u(\pi,t) = 0$$

Conclude : Need f odd and 2π-periodic.

See Stein for answer when $g \not\equiv 0$. p. 8-10.