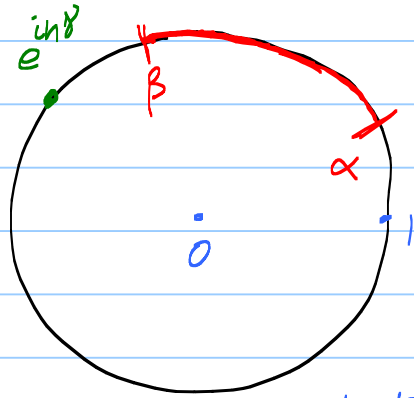
Lesson 23 Weyl's equidistribution theorem

HWK 5 due Wed March 22 after break. See hints on Piazza

Herman Weyl ("Vile")

André Weil ("Vay")



γ an irrational multiple of π. Then $e^{in\gamma}$ are equidistributed on the unit circle $C_1(0)$.

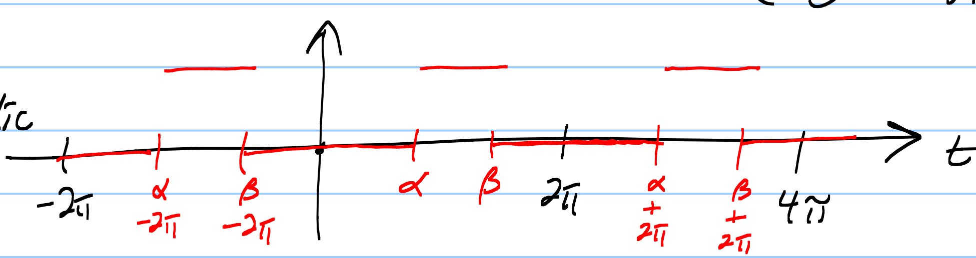
Define $$R_N = \frac{\#\text{ times } e^{in\gamma} \text{ falls in } (\alpha,\beta) \text{ arc as } n=1,2,\dots,N}{N}$$

$$\xrightarrow[\text{Thm}]{} \frac{\beta-\alpha}{2\pi} \quad \text{as } N\to\infty.$$

Pf: Define $$\chi(t) \underset{\substack{\uparrow\\ \text{notation}\\ \text{abuse}}}{=} \chi(e^{it}) = \chi_{(\alpha,\beta)} = \begin{cases} 1 & e^{it}\in\{e^{i\theta} : \alpha<\theta<\beta\} \\ 0 & \text{otherwise}\end{cases}$$

$$= \begin{cases} 1 & t\in(\alpha,\beta) \text{ modulo } 2\pi \\ 0 & \text{otherwise}\end{cases}$$

$\chi(t)$ is 2π-periodic



Notice $$R_N = \frac{\sum_{n=1}^{N}\chi(e^{in\gamma})}{N}$$ ← Aha! Looks like a Riemann sum!

$$= \frac{1}{2\pi}\cdot\underbrace{\left(\frac{2\pi}{N}\right)}_{\Delta x}\sum_{1}^{N}\chi(e^{in\gamma}) \xrightarrow{\text{hope}} \frac{1}{2\pi}\int_{-\pi}^{\pi}\chi(t)\,dt \overset{\checkmark}{=} \frac{\beta-\alpha}{2\pi}$$

Weyl's lemma: If $f(t)$ is continuous and 2π-periodic, then $\frac{1}{N}\sum_{n=1}^{N} f(e^{in\gamma}) \longrightarrow \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt$

Pf: Step 1: True for trig polys

$$A_0 + \sum_{n=1}^{N}(A_n \cos nt + B_n \sin nt) = \sum_{-N}^{N} C_N e^{int}$$

(Note: coeff don't have to the Fourier coeff)

Step 0: True for $f(t) \equiv 1$:

$$R_N = \frac{\overbrace{1+1+\cdots+1}^{N \text{ times}}}{N} = 1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1\,dt \;\checkmark$$

Step .5: True for other basis fcns for trig polys

$f_m(t) = e^{imt}$ Then $f_m(e^{in\gamma}) = e^{im(n\gamma)}$

$$= (e^{im\gamma})^n$$

$$R_N = \frac{f_m(e^{i\gamma}) + f_m(e^{i2\gamma}) + \cdots + f_m(e^{iN\gamma})}{N}$$

$$= \frac{(e^{im\gamma})^1 + (e^{im\gamma})^2 + \cdots + (e^{im\gamma})^N}{N}$$

$$= \frac{e^{im\gamma}}{N}\left(1 + (e^{im\gamma}) + \cdots + (e^{im\gamma})^{N-1}\right)$$

$$= \frac{e^{im\gamma}}{N} \frac{1-(e^{im\gamma})^N}{1-e^{im\gamma}}$$ ⟵ only place use $\gamma \neq \frac{p}{q}\pi$.

Important: $e^{i\theta} = 1$ if and only if $\theta = n(2\pi)$ for some $n \in \mathbb{Z}$.

Since γ = irrat. mult. π, $e^{im\gamma} \neq 1$.

$$\text{Now } |R_N| = \frac{\overbrace{|e^{im\gamma}|}^{1}}{N} \frac{|1 - e^{im\gamma N}|}{|1 - e^{im\gamma}|} \quad |1 - e^{im\gamma N}| \le 1 + 1 = 2$$

$$\le \frac{2}{N|1 - e^{im\gamma}|} \longrightarrow 0 \quad \text{as } N \to \infty.$$

$$\xrightarrow{?} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{imt}\, dt = 0$$

Step 0.75: True for 1, e^{imt}. Hence true for linear combos = trig polys. (By linearity)

Step 2 Use Fejér's to get Weyl's lemma.

Let $\varepsilon > 0$. $f(t)$ cont and 2π-periodic.

Fejér's says $\exists$ trig poly $P(t)$ with

$|f(t) - P(t)| < \frac{\varepsilon}{3}$ for all t.

$$\mathcal{E}_N = \left| \frac{1}{N} \sum_{n=1}^{N} f(e^{in\gamma}) - \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, dt \right|$$

$$= \Bigg| \underbrace{\frac{1}{N} \sum_{n=1}^{N} \Big(f(e^{in\gamma}) - P(e^{in\gamma}) \Big)}_{T_1} + \underbrace{\Big(\frac{1}{N} \sum_{n=1}^{N} P(e^{in\gamma}) - \frac{1}{2\pi} \int_{-\pi}^{\pi} P(t)\, dt \Big)}_{T_2}$$

$$+ \underbrace{\frac{1}{2\pi}\int_{-\pi}^{\pi}\big(P(t) - f(t)\big)\,dt}_{T_3} \Bigg|$$

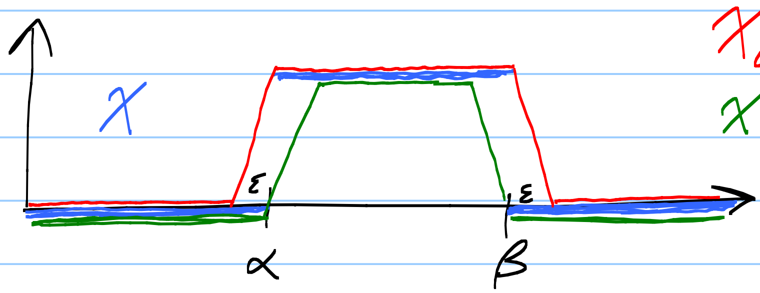
$$|T_1| \le \frac{1}{N}\Big(\underbrace{\tfrac{\varepsilon}{3} + \cdots + \tfrac{\varepsilon}{3}}_{N \text{ times}}\Big) \le \frac{\varepsilon}{3} \quad \checkmark$$

Can make $|T_2| < \frac{\varepsilon}{3}$ by taking $N >$ some N_0.

And $$|T_3| \le \frac{1}{2\pi}\int_{-\pi}^{\pi} \underbrace{|P(t) - f(t)|}_{<\frac{\varepsilon}{3}}\,dt \le \frac{1}{2\pi}\left(\frac{\varepsilon}{3}\right)\cdot 2\pi \le \frac{\varepsilon}{3} \quad \checkmark$$

So $|\mathcal{E}_N| < \varepsilon$ if $N > N_0$. Done!

Last case in Weyl's theorem proof:



χ_{over}, χ_{under} continuous

extend to be 2π-periodic

$$\chi_u \le \chi \le \chi_o$$

$$\frac{1}{N}\sum_{n=1}^{N}\chi_u(e^{in\gamma}) \le \frac{1}{N}\sum_{n=1}^{N}\chi(e^{in\gamma}) \le \frac{1}{N}\sum_{n=1}^{N}\chi_o(e^{in\gamma})$$

$N \to \infty$:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\chi_u\,dt \qquad\qquad \frac{1}{2\pi}\int_{-\pi}^{\pi}\chi_o\,dt$$

$\parallel \qquad\qquad \parallel$

$$\frac{\beta-\alpha-\varepsilon}{2\pi} \qquad\qquad \frac{\beta-\alpha+\varepsilon}{2\pi}$$

Hence $\displaystyle \frac{1}{N}\sum_{n=1}^{N} \chi(e^{in\theta}) \longrightarrow \frac{\beta-\alpha}{2\pi}$

Hmmm: We just showed lemma is true for step fcns. Fact: Riemann integrable fcns can be well approximated by step fcns. Conclude Weyl's thm for Riemann int fcns.

Computer throwing darts in $[0,1]$.

Get 2 gigantic primes p, q

Take big N from clock on computer.

"Throw dart" $\underbrace{N \cdot \frac{p}{q} \mod \mathbb{Z}}_{\text{decimal part}}$

"pseudorandom number"