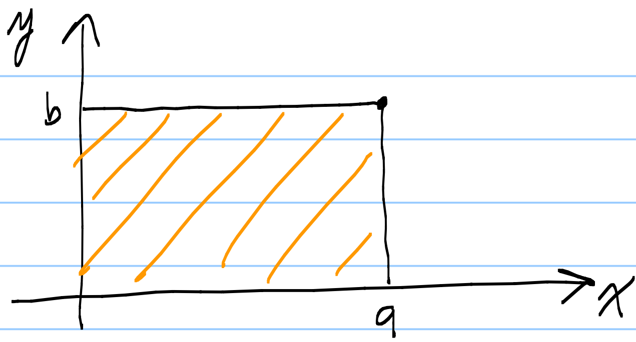
Lesson 24 The vibrating square drum problem

Hwk 5 due Wed after break



$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u$$

$u(x,y,t) = 0$ on edges

$$\begin{cases} u(x,y,0) = f(x,y) \\ \dfrac{\partial u}{\partial t}(x,y,0) = g(x,y) \end{cases}$$

Try $u(x,y,t) = X(x)Y(y)T(t)$

$$XYT'' \overset{\text{want}}{=} c^2\left(X''YT + XY''T\right)$$

Take $c=1$. Divide by XYT

$$\boxed{\frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y}} = \lambda \text{, a constant}$$

(Let $t=\frac{1}{2}$, Let (x,y) pt in drum, etc.)

T-problem: $T'' - \lambda T = 0$.

X,Y - problem: $\dfrac{X''}{X} + \dfrac{Y''}{Y} = \lambda$

$$\frac{X''}{X} = \lambda - \frac{Y''}{Y} = \mu \text{, a const,}$$

X-prob: $X'' - \mu X = 0$ BC: $X(0)=0$, $X(a)=0$

Y-prob: $Y'' - (\lambda - \mu)Y = 0$ BC: $Y(0)=0$, $Y(b)=0$

Know sol^n to X-prob: To get non-zero sol^{n}s, need

$$\mu_n = -\left(\frac{n\pi}{a}\right)^2, \quad X_n(x) = \sin\frac{n\pi x}{a}$$

Y-prob: $\quad Y'' - (\lambda - \mu_n)Y = 0\ ; \quad Y(0)=0,\ Y(b)=0$

Only get non-zero sol^{n}s when

$$\boxed{\lambda - \mu_n = -\left(\frac{mb}{\pi}\right)^2}, \quad Y_m(y) = \sin\frac{m\pi y}{b}$$

T-prob $\quad T'' - \lambda T = 0 \qquad \lambda = -\left(\frac{mb}{\pi}\right)^2 + \mu_n$

$$r^2 + \left[\left(\frac{mb}{\pi}\right)^2 + \left(\frac{na}{\pi}\right)^2\right] = 0 \qquad \boxed{\lambda = -\left(\frac{mb}{\pi}\right)^2 - \left(\frac{na}{\pi}\right)^2}$$

So $\quad T_{nm}(t) = A_{nm}\cos\omega_{nm}t + B_{nm}\sin\omega_{nm}t$

where $\quad \omega_{nm} = \sqrt{\left(\frac{mb}{\pi}\right)^2 + \left(\frac{na}{\pi}\right)^2} \qquad r = \pm\,\omega_{nm}\,i$

Basic sol^n $\quad u_{nm}(x,y,t) = X_n(x)Y_m(y)T_{nm}(t)$

$$= \sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}\left(A_{nm}\cos\omega_{nm}t + B_{nm}\sin\omega_{nm}t\right)$$

Gen^l sol^n: $\quad u(x,y) = \sum u_{nm}(x,y,t)$

Last thing to do: I.C.:

$$(*) \quad u(x,y,0) = \sum_{n,m=1}^{\infty} A_{nm}\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b} \overset{\text{want}}{=} f(x,y)$$

Invent Fourier sine series in <u>two</u> variables!

$$\frac{\partial u}{\partial t}(x,y,t) = \sum_{n,m=1}^{\infty} \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b}\Big[\omega_{nm} A_{nm}(-\sin \omega_{nm} t) + \omega_{nm} B_{nm} \cos \omega_{nm} t\Big]$$

(*) $$\frac{\partial u}{\partial t}(x,y,0) = \sum_{n,m=1}^{\infty} \omega_{nm} B_{nm} \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b} \overset{\text{want}}{=} g(x,y)$$

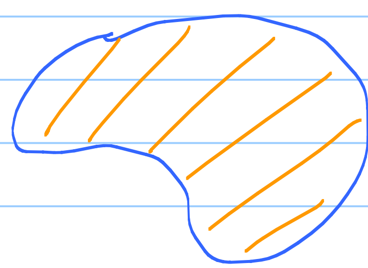
Frequency of basic sol^{n}s : $\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}$ Ugh! ugly sound

Pitch of drum: "Earliest" term in overtones (Lowest freq of basic sol^{n}s)

Pitch : $\sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2}$

Fun question: Which drum of area $A = ab$ has lowest pitch? Answer: Square! $a = b = \sqrt{A}$!

Famous problem Can you hear the shape of a drum?



Ω

$$\frac{\partial^2 u}{\partial t^2} = c^2 \Delta u$$

Eigenfunctions of Laplacian: $\Delta \varphi = \lambda \varphi$, $\varphi = 0$ on edge

$\lambda_1, \lambda_2, \lambda_3, \dots$ determine pitches of overtones

Do they determine Ω? λ's : spectrum

Carolyn Gordon: No!

Circular drum problem: Laplacian in polar coord.

$$u(r,\theta,t) = \underbrace{R(r)}\,\Theta(\theta)\,T(t)$$

R fcns satisfy Bessel eqns!

λ's related to zeroes of Bessel fcns!

Still not a pretty sound.

Timpani: The shape of the volume of air
in the kettle drum plays a role.