Lesson 26 The real Fourier transform

HWK 5 due tonight 11pm in Gradescope

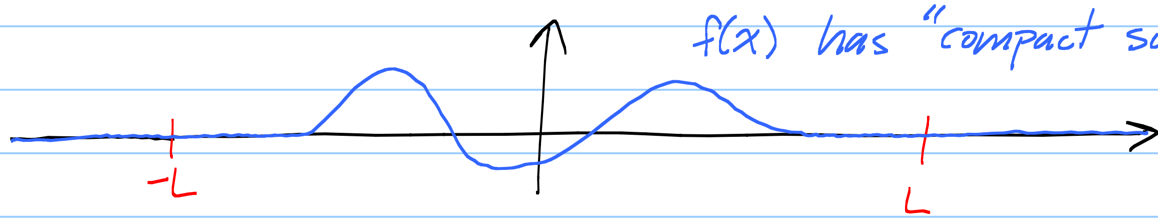
Complex Fourier transform: $\hat{f}(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$

Mind blowing fact: The Fourier inversion formula:

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(s)\, e^{isx}\, ds$$

Think $\displaystyle f(x) \approx \sum_{w_n \in \mathbb{R}} \hat{f}(w_n)\, e^{i w_n x}\, \Delta w$ ← expand f in trig fcns of all the basic frequencies

Motivation for the real version of the F.T.



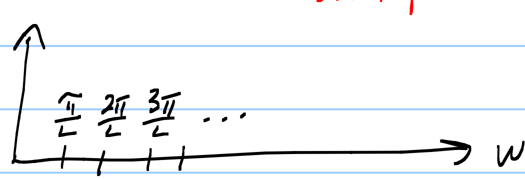
$f(x)$ has "compact support" and is C^2-smooth

Idea: $\displaystyle f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$ Let $L \to \infty$

$$= \underbrace{\left(\frac{1}{2L}\int_{-L}^{L} f(t)\,dt\right)}_{a_0 \,\to\, 0 \text{ as } L\to\infty} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\left(\underbrace{\underbrace{\frac{1\cdot\pi}{L}}_{\Delta w}\int_{-L}^{L} f(t)\cos \underbrace{\frac{n\pi}{L}}_{w_n} t\, dt}_{a_n}\right)\cos\underbrace{\frac{n\pi}{L}}_{w_n} x + \cdots \sin \cdots \sin\right]$$

$$= \underset{\text{sum}}{\text{Riemann}} = \sum_{1}^{\infty} \frac{1}{\pi}\left[\left(\int_{-\infty}^{\infty} f(t)\cos w_n t\, dt\, \cos w_n x\right)\Delta w + (\sin \text{ stuff})\right]$$

↑ Can replace L by ∞ because f has compact support



$$\xrightarrow[L\to\infty]{} \int_0^\infty A(w)\cos wx + B(w)\sin wx \, dw$$

$$\text{where} \begin{cases} A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos wt \, dt \\ B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin wt \, dt \end{cases}$$

Can be made rigorous! Summary:

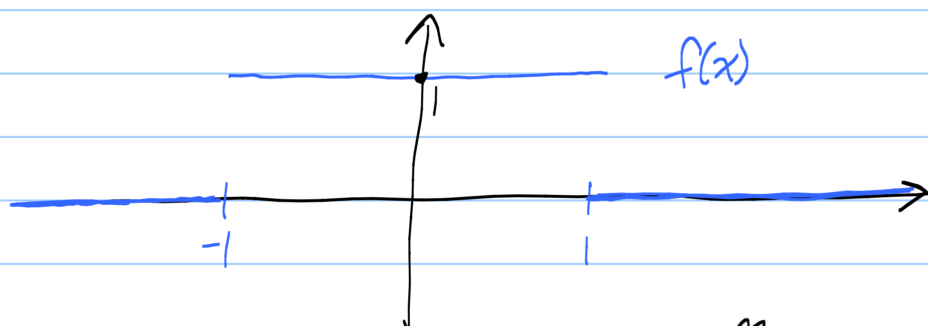
The "Fourier integral"

$$\left[\begin{array}{l} f(x) = \int_0^\infty A(w)\cos wx + B(w)\sin wx \, dw \\ \text{where} \begin{cases} A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos wt \, dt \\ B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin wt \, dt \end{cases} \end{array}\right]$$

<u>Fact</u>: If f is piecewise C^1-smooth and $\int_{-\infty}^{\infty} |f| \, dx$ is finite,

then $f(x)$ "=" its Fourier integral $= \begin{cases} f(x) & \text{where } f \text{ cont.} \\ \text{midpoint of jumps} \end{cases}$

<u>EX</u>:



$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} \underbrace{f(t)}_{\text{even}} \underbrace{\cos wt}_{\text{even}} \, dt = \frac{2}{\pi}\int_0^\infty f(t)\cos wt \, dt$$

$$= \frac{2}{\pi}\int_0^1 1\cdot\cos wt \, dt$$

$$= \frac{2}{\pi}\left[\frac{1}{w}\sin wt\right]_{t=0}^{1}$$

$$= \frac{2}{\pi}\,\frac{\sin w}{w} \quad = A(w)$$

$$B(w) = \frac{2}{\pi}\int_{-\infty}^{\infty} \underbrace{\underbrace{f(t)}_{\text{even}}\ \underbrace{\sin wt}_{\text{odd}}}_{\text{odd}}\, dt \quad = 0$$

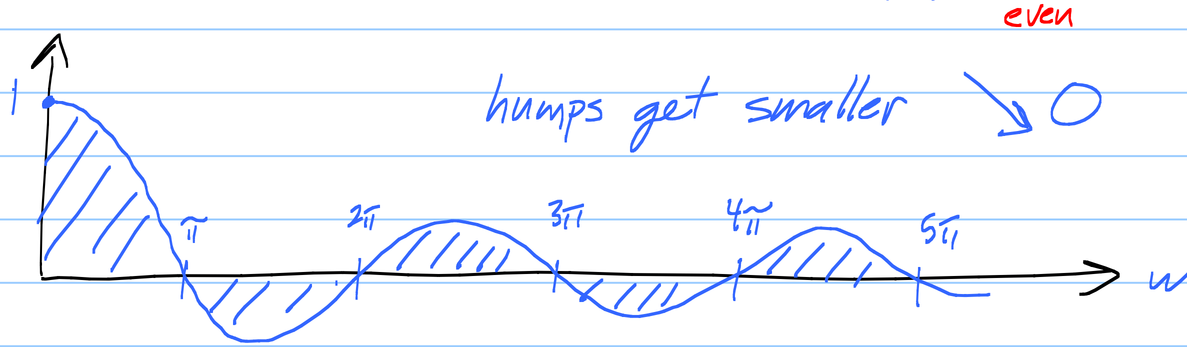
$$f(x) \overset{\text{""}}{=} \int_0^\infty A(w)\cos wx + B(w)\sin wx \quad dw$$

$$= \boxed{\int_0^\infty \frac{2}{\pi}\,\frac{\sin w}{w}\cos wx \quad dw}$$

$$= \begin{cases} 0 & x < -1 \\ \frac{1}{2} & x = -1 \\ 1 & -1 < x < 1 \\ \frac{1}{2} & x = 1 \\ 0 & x > 1 \end{cases}$$

Whoa! Let $x=0$. Get $1 = \frac{2}{\pi}\int_0^\infty \frac{\sin w}{w}\, dw$

Famous integral: $\int_0^\infty \frac{\sin w}{w}\, dw = \frac{\pi}{2}$ $\left(\int_{-\infty}^{\infty} \underbrace{\frac{\sin w}{w}}_{\text{even}}\, dw = \pi\right)$

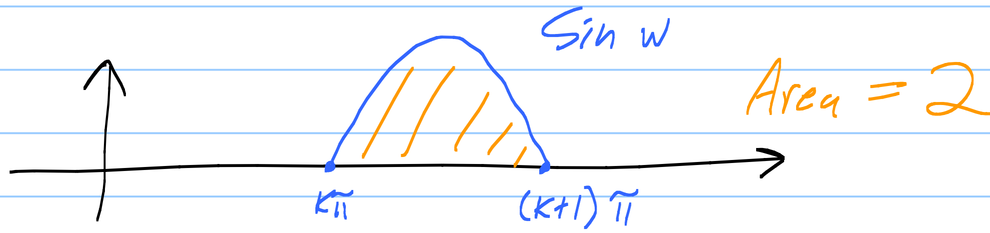


humps get smaller $\searrow 0$

Alternating series test: $\lim_{L\to\infty} \int_0^L \frac{\sin w}{w}\, dw$ exists.

However $\int_0^\infty \left|\frac{\sin w}{w}\right| dw = \infty$

$\int$ is conditionally convergent



Integrate $\left|\frac{\sin w}{w}\right|$ from $k\pi$ to $(k+1)\pi$:

$$\frac{|\sin w|}{(k+1)\pi} \le \left|\frac{\sin w}{w}\right| \le \frac{|\sin w|}{k\pi}$$

$$\int_{k\pi}^{(k+1)\pi}: \quad \frac{2}{(k+1)\pi} \le \int_{k\pi}^{(k+1)\pi} \left|\frac{\sin w}{w}\right| dw \le \frac{2}{k\pi}$$

$$\sum_{k=0}^{\infty} = \infty$$

Harmonic series

p-series $p=1$, Diverges!

Next time: Think about $f(x)$ being even and odd.

Get Fourier sine transform, and cosine transform.