Lesson 27 Fourier Cosine & Sine Transforms

Hwk 6 due Mon, April 3

Fourier integral: $$f(x) = \int_0^\infty A(w)\cos wx + B(w)\sin wx \, dw$$

where $$\begin{cases} A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos wx \, dx \\ B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\sin wx \, dx \end{cases}$$

What if f is even? $$\begin{cases} A(w) = \frac{2}{\pi}\int_0^{\infty} f(x)\cos wx \, dx \\ B(w) = 0 \end{cases}$$

Aha! Given f on $[0,\infty)$, extend as an even fcn.

See $$\left[\begin{array}{l} f(x) = \int_0^\infty A(w)\cos wx \, dw \\ \quad \text{where } A(w) = \frac{2}{\pi}\int_0^\infty f(x)\cos wx \, dx \end{array}\right.$$

The Fourier Cosine transform: $$\tilde{\mathcal{F}}_c[f] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos wx \, dx$$

Great thing! $\boxed{\tilde{\mathcal{F}}_c \circ \tilde{\mathcal{F}}_c = \text{identity}}$

Sends fcns on $[0,\infty)$ to fcns on $[0,\infty)$.

Similarly, Fourier Sine transform is $$\tilde{\mathcal{F}}_s[f] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin wx \, dx$$

$$\boxed{\tilde{\mathcal{F}}_s[\tilde{\mathcal{F}}_s[f]] = f}$$

To undo $\tilde{\mathcal{F}}_c$ or $\tilde{\mathcal{F}}_s$, just do it again!

Other key properties

$$\begin{cases} \tilde{\mathcal{F}}_c[f'] = w\,\tilde{\mathcal{F}}_s[f] - \sqrt{\frac{2}{\pi}}\,f(0) \\ \tilde{\mathcal{F}}_s[f'] = -w\,\tilde{\mathcal{F}}_c[f] \end{cases}$$

$\leftarrow$ need $f(x) \to 0$ as $x \to \infty$

Repeat:

$$\begin{cases} \tilde{\mathcal{F}}_c[f''] = -w^2\,\tilde{\mathcal{F}}_c[f] - \sqrt{\frac{2}{\pi}}\,f'(0) \\ \tilde{\mathcal{F}}_s[f''] = -w^2\,\tilde{\mathcal{F}}_s[f] + \sqrt{\frac{2}{\pi}}\,f(0)\,w \end{cases}$$

Why: $\tilde{\mathcal{F}}_c[f'](w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x)\,\underbrace{\cos wx}_{u}\,dx$

$dv = f'(x)\,dx \qquad du = -w\sin wx\,dx$

$v = f(x)$

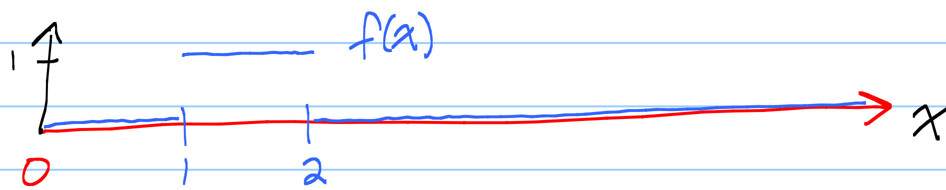
$$= \lim_{L\to\infty} \sqrt{\frac{2}{\pi}} \left[\underbrace{(\cos wx)\,f(x)\Big|_0^L}_{0 - f(0)} - \underbrace{\int_0^L f(x)\,[-w\sin wx]\,dx}_{w\int_0^\infty f(x)\sin wx\,dx} \right]$$

assuming $f(L) \to 0$ as $L \to \infty$

$$= w\,\tilde{\mathcal{F}}_s[f] - \sqrt{\frac{2}{\pi}}\,f(0) \quad \checkmark$$

Similarly for $\tilde{\mathcal{F}}_s[f']$. (Easier because $\sin 0 = 0$)

Application Semi-infinite hot wire problem



Heat eqn: $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ (take $c=1$)

Initial cond: $u(x,0) = f(x)$

Boundary cond: $u(0,t)=0$ ← refrigerated at $x=0$.

Big idea: Take $\tilde{\mathcal{F}}_s$ in the space variable x.

[reason for $\tilde{\mathcal{F}}_s$ instead of $\tilde{\mathcal{F}}_c$: $f(0)$ term on right instead of $f'(0)$]

Write $$\hat{u}(w,t) = \sqrt{\tfrac{2}{\pi}} \int_0^\infty u(x,t) \sin wx \, dx$$

PDE?
$$\frac{\partial \hat{u}}{\partial t} = \sqrt{\tfrac{2}{\pi}} \int_0^\infty \underbrace{\frac{\partial u}{\partial t}(x,t)}_{\frac{\partial^2 u}{\partial x^2}(x,t)} \sin wx \, dx$$

$$= \tilde{\mathcal{F}}_s\left[\frac{\partial^2 u}{\partial x^2}\right]$$

$$= -w^2 \underbrace{\tilde{\mathcal{F}}_s[u]}_{\hat{u}} + \sqrt{\tfrac{2}{\pi}}\, \underbrace{u(0,t)\, w}_{=0}$$

Aha! $$\boxed{\frac{\partial \hat{u}}{\partial t} = -w^2 \hat{u}}$$ ← easy!

Solve it: $$\hat{u} = \underbrace{C(w)}_{\text{this const might depend on } w.} e^{-w^2 t}$$

Remark: $\frac{\partial}{\partial t}\left[\frac{\hat{u}}{e^{-w^2 t}}\right] \equiv 0$. So $\frac{\hat{u}}{e^{-w^2 t}}$ is a fcn of w.

Next Determine $C(w)$ from initial cond: $u(x,0)=f(x)$

$$\hat{u}(w,0) = \sqrt{\tfrac{2}{\pi}} \int_0^\infty \underbrace{u(x,0)}_{f(x)} \sin wx \, dx = \sqrt{\tfrac{2}{\pi}} \int_1^2 1 \cdot \sin wx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[-\frac{1}{w} \cos wx \right]_1^2 = \sqrt{\frac{2}{\pi}} \left(\frac{\cos w - \cos 2w}{w} \right) = C(w)$$

Recall: $\hat{u} = C(w) e^{-w^2 t}$

Plug in $t=0$: $\hat{u}(w,0) = C(w) \cdot 1$ ✓

We have $\hat{u}(w,t) = \sqrt{\frac{2}{\pi}} \left(\frac{\cos w - \cos 2w}{w} \right) e^{-w^2 t}$

Last step: Hit it with $\tilde{\mathcal{F}}_s$ to undo $\tilde{\mathcal{F}}_s$!

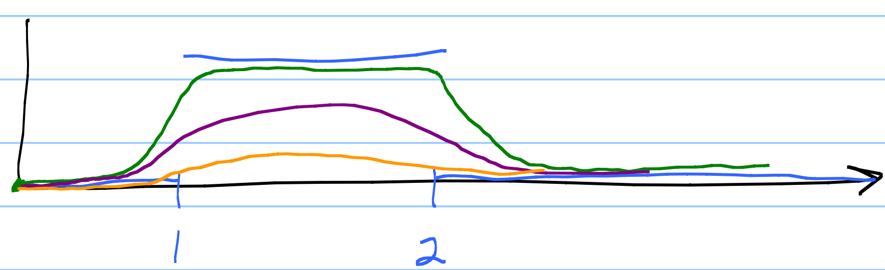
Ans: $$\boxed{u(x,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left(\frac{\cos w - \cos 2w}{w} \right) e^{-w^2 t} \sin wx \, dw}$$

$\int_0^\infty$ converges very fast because of $e^{-w^2 t}$ term.

$$\underbrace{\frac{\cos w - \cos 2w}{w}}_{\text{nice at } w=0} = \frac{\left(1 - \frac{w^2}{2!} + \frac{w^4}{4!} - \cdots\right) - \left(1 - \frac{4w^2}{2!} + \frac{16w^4}{4!} - \cdots\right)}{w}$$

$= \frac{3}{2!} w +$ higher order terms   R of C $= \infty$

(or use L'H rule).



Formula shows that $u(x,t) \to 0$ as $t \to \infty$ fast!

Case: $\underbrace{\text{Wire insulated at } x=0}_{\frac{\partial u}{\partial x}(0,t) = 0}$: Use $\tilde{\mathcal{F}}_c$ instead of $\tilde{\mathcal{F}}_s$.