Lesson 30 Dirichlet problem on the upper half plane

HWK 6 due Mon

Heisenberg uncertainty principle: Finishing touches

Reverse implication: $\hat{f}$ compact support ($\not\equiv 0$) $\Rightarrow$ f cannot have compact supp

Easy: $\mathcal{F}^{-1}[g] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(s)\, e^{isx}\, ds$

$g = \mathcal{F}[f]$

Same proof as other direction: $|s| = \frac{n\pi}{L}$ $\quad n > N$

Case: f merely continuous with compact support.

Problem: Don't know F.S. $\to$ f at <u>any</u> pt.

Even so, we can see f must be a trig poly on $[-L, L]$.

<u>Why</u>: Parseval's: Have $\frac{1}{2L}\int_{-L}^{L} f(x)\, e^{-inx}\, dx = 0 \quad |n| \geq N.$

So Fourier coeff $c_n = 0$ when $|n| > N$.

Idea: $F(x) = f(x) - \sum_{-N}^{N} c_n e^{i\frac{n\pi}{L}x}$

Aha! F is cont and <u>all</u> its F. coeff A_n are zero!

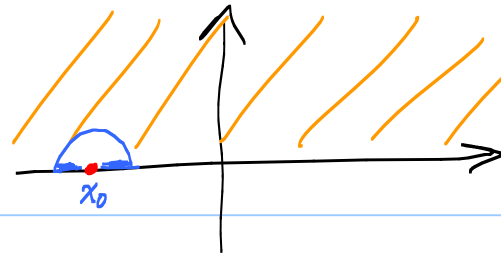
Parseval's: $\frac{1}{2L}\int_{-L}^{L} F^2\, dx = \sum_{-\infty}^{\infty} |A_n|^2 = 0$.

F cont with $\int_{-L}^{L} F^2\, dx = 0 \Rightarrow F \equiv 0$ on $[-L, L]$.

Me: Fejér's $\Rightarrow$ Parseval's $\Rightarrow$ uniqueness of F. coeff

Stein: Fejér's $\Rightarrow$ uniq of F. coeff $\Rightarrow$ Parseval's

D. Prob on the UHP



UHP

Given continuous fcn $f(x)$ on $\mathbb{R}$ with $\lim_{x \to \pm\infty} f(x) = 0$.

Find a harmonic fcn $u(x,y)$ on UHP that is continuous up to $\mathbb{R}$ such that $u(x,0) = f(x)$.

Pf: $\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} \equiv 0 \\ u(x,0) = f(x) \end{cases}$ ← Steady state temp

Try $u(x,y) = X(x)Y(y)$

$$X''Y + XY'' = 0$$

$$\frac{X''}{X} = \frac{-Y''}{Y} = \lambda$$

$$\begin{cases} X'' - \lambda X = 0 \\ Y'' + \lambda Y = 0 \end{cases}$$

Case $\lambda = k^2 > 0$. $r^2 - k^2 = 0$, $r = \pm k$

$X(x)$ sol^{n}s: e^{kx}, e^{-kx}

$\to \infty$ as $x \to \infty$ $\qquad \to \infty$ as $x \to -\infty$

ouch!

Can't allow our sol^{n}s to get hotter than hell!

Case $\lambda = 0$: $X(x) = Ax + B$. Need $A = 0$.

Get const. sol^n: $X(x) = B$.

Case $\lambda = -k^2 < 0$: $r = \pm ki$

$$X(x) = c_1 \cos kx + c_2 \sin kx$$

Exciting idea: Today $X(x) = A_1 e^{ikx} + A_2 e^{-ikx}$

Get sol^{n}s $X(x) = e^{isx}$ $\quad s = k, -k$

Y-problem: $Y'' - k^2 Y = 0$ $\quad r = \pm k$

$$Y(y) = c_1 \underbrace{e^{ky}}_{\text{ouch}} + c_2 e^{-ky}$$

$\to \infty$
$y \to \infty$

Need $c_1 = 0$.

Found sol^{n}s

$$u(x,y) = X(x)Y(y) = e^{isx} e^{-|s|y} \qquad s \in \mathbb{R}$$

Hmmm. Get lot's of sol^{n}s $\sum c_n e^{is_n x} e^{-|s_n| y}$

Aha! Think Riemann sum!

Try $\quad u(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(s) e^{isx} e^{-|s|y} \, ds$

If we assume $\int$ converges nicely, can move Δ

under the $\int$ to see $u(x,y)$ is harmonic on UHP.

Last thing: Want $u(x,0) = f(x)$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(s) e^{isx} \, ds \overset{\text{want}}{=} f(x)$$

Aha! $\mathcal{F}^{-1}[g] = f$. $\quad$ Need $g(s) = \hat{f}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-isx} \, dx$

Sol^n:

$$\boxed{u(x,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(s)\,e^{isx}e^{-|s|y}\,ds}$$

Poisson Kernel for the UHP

$$u(x,y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(t)\,e^{-ist}\,dt\right)e^{isx}e^{-|s|y}\,ds$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(t)\underbrace{\left[\int_{-\infty}^{\infty}e^{is(x-t)-|s|y}\,ds\right]}_{P(x-t,\,y)}dt$$

Compute it! $\displaystyle P(x-t,y)=\int_{-\infty}^{0}+\int_{0}^{\infty}$

$$\underbrace{\int_{-\infty}^{0}}_{s<0} \quad + \quad \underbrace{\int_{0}^{\infty}}_{s>0}$$

$|s|=-s$

$\vdots$

$$=\frac{1}{i(x-t)+y}$$

$$=\frac{1}{i(x-t)-y}\,e^{[i(x-t)-y]s}\Big|_{0}^{\infty}$$

($-ys$ makes $\to 0$ as $s\to\infty$)

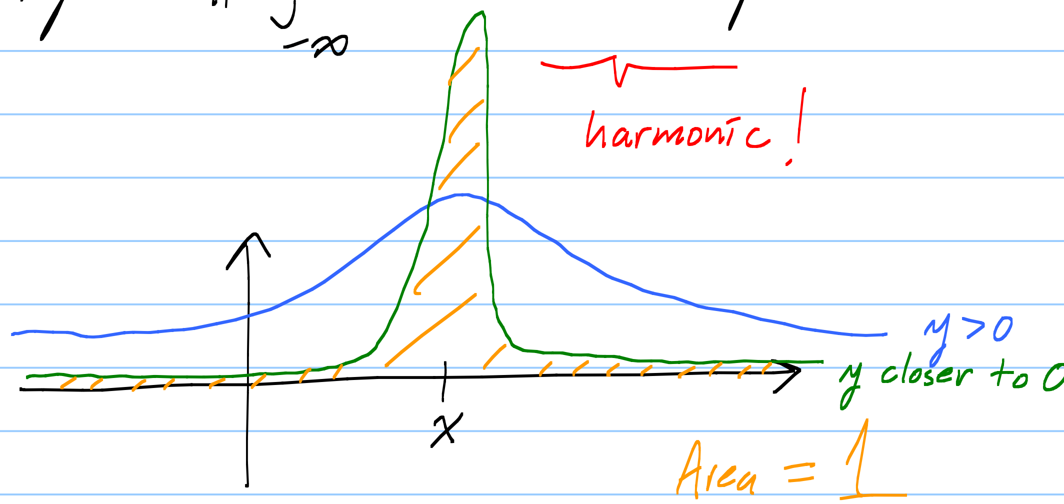
$$=0-\frac{1}{i(x-t)-y}$$

$$P(x-t,y)=\frac{2y}{(x-t)^2+y^2}$$

Poisson formula for UHP

$$u(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\, \underbrace{\frac{y}{(x-t)^2+y^2}}_{\text{harmonic!}}\, dt$$

Good kernel!



Use good kernel ideas to show $u(x,y) \to f(x_0)$

as $(x,y) \to (x_0, 0)$ in the UHP.