Lesson 33 Fourier inversion!

Practice problems for Exam 2 on home page

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(s) e^{i2\pi sx} \, ds \quad \text{where} \quad \hat{f}(s) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi sx} \, dx$$

for $f \in \mathcal{S}$: $\sup_{x\in\mathbb{R}} |f^{(n)}(x) x^m| < \infty$ for each m, n

(Same as $f^{(n)}(x) x^m \to 0$ as $x \to \infty$, $x \to -\infty$ $\forall m, n$)

Loose end from Lesson 32. $\varphi(x) = e^{-\pi x^2}$ "bump fcn"

Forgot to mention $\hat{\varphi}(s)$ is real valued

$$\hat{\varphi}(s) = \int_{-\infty}^{\infty} \underbrace{e^{-\pi x^2}}_{\text{even}} \; \underbrace{e^{-i2\pi sx}}_{\underbrace{\cos(2\pi sx)}_{\text{even}} - i \underbrace{\sin(2\pi sx)}_{\text{odd}}} \, dx$$

$$= 2\int_0^{\infty} e^{-\pi x^2} \cos(2\pi sx) \, dx \quad \checkmark$$

Real ODE argument valid: $\hat{\varphi}(s) = e^{-\pi s^2}$

Note: $|f(x)x^2| < M \implies |f(x)| < \frac{M}{x^2}$ for $|x| > 1$.

So $\int_{-\infty}^{\infty} |f| \, dx < \infty$.

Fact: $\mathcal{F}[f] = \hat{f}$. $\mathcal{F} : \mathcal{S} \to \mathcal{S}$

Why: $|\hat{f}(s)| = \left| \int_{-\infty}^{\infty} f(x) \underbrace{e^{-i2\pi sx}}_{\text{modulus} = 1} \, dx \right| \le \int_{-\infty}^{\infty} |f(x)| \cdot 1 \, dx < \infty$

So $\hat{f}$ is bndd.

Key: Facts $\mathcal{F}[x f(x)] = c \hat{f}'(s)$ $c = \frac{1}{-2\pi i}$

$\mathcal{F}[f'(x)] = 2\pi i s \hat{f}(s)$

Repeat arg above to see $\hat{f}^{(n)}(s) s^m$ bndd $\forall m, n$

Lemma: $f \in \mathcal{S}$. Then $\varphi_\varepsilon * f \to f$ uniformly on $\mathbb{R}$.

[So $(\varphi_\varepsilon * f)(0) \to f(0) \leftarrow$ all we need]

Pf: "Good Kernel" argument:

$$(\varphi_\varepsilon * f)(x) - f(x) = \underbrace{\int_{-\infty}^{\infty} \varphi_\varepsilon(x-t) f(t)\, dt}_{\int_{-\infty}^{\infty} \varphi_\varepsilon(t) f(x-t)\, dt} - f(x) \int_{-\infty}^{\infty} \varphi_\varepsilon(t)\, dt$$

$$= \int_{-\infty}^{\infty} \varphi_\varepsilon(t) \left[f(x-t) - f(x) \right] dt$$

etc, ...

Lemma $f, g \in \mathcal{S}$: $\int_{-\infty}^{\infty} f(x) \hat{g}(x)\, dx = \int_{-\infty}^{\infty} \hat{f}(x) g(x)\, dx$

Pf: Fubinni: on $f(x) g(y) e^{-i 2\pi x y}$

$$\int_{-\infty}^{\infty} f(x) \underbrace{\left(\int_{-\infty}^{\infty} g(y) e^{-i 2\pi x y}\, dy \right)}_{\hat{g}(x)} dx$$

$$\underset{\text{Fub}}{=} \int_{-\infty}^{\infty} g(y) \underbrace{\int_{-\infty}^{\infty} f(x)\, e^{-i2\pi xy}\, dx}_{\hat{f}(y)}\, dy$$

Last: change dummy variable from y to x here. ✓

Fourier inversion theorem: $f(x) = \int_{-\infty}^{\infty} \hat{f}(s)\, e^{i2\pi sx}\, ds$

Pf: Step 1: True for $x=0$:

$$f(0) = \int_{-\infty}^{\infty} \hat{f}(s)\, ds$$

$$\int_{-\infty}^{\infty} f(x) \underbrace{\varphi_\varepsilon(x)}_{\text{want } = \hat{g}}\, dx = (\varphi_\varepsilon * f)(0) \xrightarrow[\varepsilon \to 0]{} f(0)$$

Easy to guess g: $\quad \widehat{\varphi(\varepsilon x)} = \underbrace{\frac{1}{\varepsilon}\hat{\varphi}(s/\varepsilon)}_{\varphi_\varepsilon}$

So $\quad g(x) = \varphi(\varepsilon x) = e^{-\pi \varepsilon^2 x^2}$

$$\underset{\text{as } \varepsilon \to 0}{f(0)} \leftarrow \int_{-\infty}^{\infty} f(x)\,\varphi_\varepsilon(x)\,dx = \int_{-\infty}^{\infty} f(x)\,\hat{g}(x)\,dx \overset{\text{Lemma}}{=} \int_{-\infty}^{\infty} \hat{f}(x)\, g(x)\, dx$$

$$= \underbrace{\int_{-\infty}^{\infty} \hat{f}(x)\, e^{-\pi\varepsilon^2 x^2}\, dx}$$

Claim: $\quad \longrightarrow \int_{-\infty}^{\infty} \hat{f}(x)\, dx$

Why: $\displaystyle\int_{-\infty}^{\infty} = \int_{-R}^{R} \hat{f}(x) \underbrace{e^{-\pi\varepsilon^2 x^2}}_{\substack{\to 1 \\ \text{unif} \\ \text{as } \varepsilon\to 0 \\ \text{on } [-R,R]}} dx + \underbrace{\int_{|x|>R} \cdots dx}_{\text{can make very small}}$

$\displaystyle\to \int_{-R}^{R} \hat{f}(x)\,dx \approx \int_{-\infty}^{\infty}$

Philosophy: Enforces: true on bump fcns $\Rightarrow$ true on $\mathcal{S}$.

Finally, show (true for $x=0$) $\Rightarrow$ true for any x.

Key: $\mathcal{F}[f(\cdot+h)] = e^{2\pi i s h}\hat{f}(s)$

Let $F(y) = f(y+x)$

$$f(x) = F(0) = \int_{-\infty}^{\infty} \hat{F}(s)\,ds = \int_{-\infty}^{\infty} e^{2\pi i s x}\hat{f}(s)\,ds$$

Done!

Cor: $\mathcal{F} : \mathcal{S} \xrightarrow[\text{onto}]{1\text{-}1} \mathcal{S}$

Cor: Plancherel identity: $\|f\| = \|\hat{f}\|$, $f \in \mathcal{S}$

where $\displaystyle\|g\| = \left(\int_{-\infty}^{\infty} |g(x)|^2\,dx\right)^{1/2} \leftarrow L^2$ norm on $\mathbb{R}$

$\sim$ Parseval's for F.T.

Pf: $\displaystyle\int_{-\infty}^{\infty} f\,\hat{g}\,dx = \int_{-\infty}^{\infty} \hat{f}\,g\,dx$

Hmmm. What if $\hat{g} = \bar{f}$?

Then $$g(s) = \check{\bar{f}} = \int_{-\infty}^{\infty} \overline{f(x)} \, \underbrace{e^{i2\pi s x}}_{\overline{e^{-i2\pi s x}}} \, dx$$

$$= \overline{\int_{-\infty}^{\infty} f(x)\, e^{-i2\pi s x} \, dx}$$

$$= \overline{\hat{f}(s)}$$

Get $$\int_{-\infty}^{\infty} f(x) \underbrace{\overline{f(x)}}_{\hat{g}} \, dx = \int_{-\infty}^{\infty} \hat{f}(s) \underbrace{\overline{\hat{f}(s)}}_{g} \, ds$$

Huge consequence: Can extend $\mathcal{F}$ from $\mathcal{S}$ to $L^2(\mathbb{R})$!

Remark: $\underbrace{C_0^\infty(\mathbb{R})}_{C^\infty,\ \text{compactly supported}} \subset \mathcal{S} \subset L^2(\mathbb{R})$

dense subset of $L^2(\mathbb{R})$

Given $f \in L^2(\mathbb{R})$, take seq $\varphi_n \in C_0^\infty(\mathbb{R})$, $\varphi_n \xrightarrow{L^2} f$

Show $\hat{\varphi}_n \to$ something, F.

Define $F = \hat{f}$. See $\|f\|^2 = \|\hat{f}\|^2$.

Remark: Easy to show Fourier inversion on $\mathcal{S}$ $\Rightarrow$ on $\mathcal{M}(\mathbb{R})$