Review for Exam 2 take-home exam

Exam 2 posted in Gradescope and on home page at 1:00pm today.

Due in Gradescope on Monday, April 17 at 11:00pm.

Exam 2 will be a take home exam during the second week of Apirl similar to a homework set. It will be open Stein and Shakarchi, open course notes, and access to the home page and the AFTERMATH, but no other resources. In particular, no Wolfram alpha, MAPLE, or any other programs and no calculators. Also, no social media like Piazza or other collaboration sites.

The exam will be released in Gradescope at 1:00 pm on Wednesday, April 12 It will be due in Gradescope at 11:00 pm on Monday, April 17. The exam will also be posted on the home page.

The Final Exam is

Wednesday, May 3, 7:00-9:00 pm in SCHM 313

1. Assume that $g(x)$ is a real valued C^1-smooth function on the whole real line such that $\int_{-\infty}^{\infty} |g(x)|\, dx$ is finite. Show that

$$\alpha(w) = \int_{-\infty}^{\infty} g(x) \sin wx \, dx$$

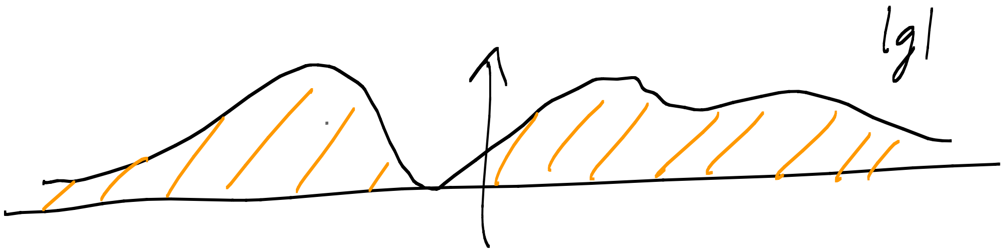
goes to zero as $w \to \infty$. Hint: Do integration by parts on a finite piece of the integral after showing that the part of the integral outside $[-N, N]$ goes to zero as $N \to \infty$.

Step 1: Let $\varepsilon > 0$. Pick $N > 0$ so large that

$$\left| \int_{|x|>N} g(x) \sin wx \, dx \right| \le \int_{|x|>N} |g(x)|\, dx < \frac{\varepsilon}{2}$$

Step 2:

$$\int_{-N}^{N} \underbrace{g(x)}_{u} \, \underbrace{\sin wx \, dx}_{dv} \qquad V = -\frac{1}{w} \cos wx$$



area $< \infty$

$$\int_{-\infty}^{\infty} |g|\, dx = \lim_{N \to \infty} \underbrace{\int_{-N}^{N} |g|\, dx}_{\substack{\text{non-decreasing} \\ \text{bounded above}}}$$

So limit exists.

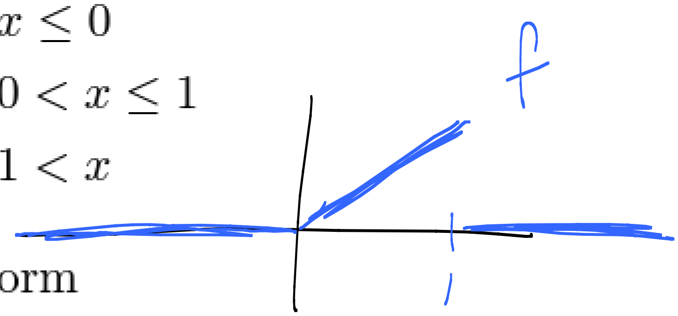
$$\int_{|x|>N} |g|\, dx = (\text{whole area}) - \int_{-N}^{N} |g|\, dx \longrightarrow 0$$

$$\left(g(x) \frac{-1}{w} \cos wx \right) \Big|_{-N}^{N} - \int_{-N}^{N} \underbrace{g'(x)}_{\text{cont fcn on } [-N,N].} \left(-\frac{1}{w} \cos wx \right) dx < \frac{\varepsilon}{2}$$

with w big

2. Define a function $f(x)$ to be used in problems 2-4 as follows:

$$f(x) = \begin{cases} 0 & x \le 0 \\ x & 0 < x \le 1 \\ 0 & 1 < x \end{cases}$$



Compute the complex Fourier transform

$$\hat{f}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$$

and the Fourier sine transform

$$[\mathcal{F}_s f](w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin wx \, dx.$$

$$\hat{f}(s) = \frac{1}{\sqrt{2\pi}} \int_0^1 x \underbrace{e^{-isx}}_{\cos(sx) - i\sin(sx)} dx \qquad = \int_0^1 + i \int_0^1$$

Int by parts: $u = x \qquad dv = e^{-isx}\, dx$

$du = dx \qquad v = \frac{-1}{is} e^{-isx}$

$$= \frac{1}{\sqrt{2\pi}} \left[x\left(-\frac{1}{is} e^{-isx}\right)\Big|_0^1 - \int_0^1 \left(-\frac{1}{is}\right) e^{-isx}\, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{-e^{-is}}{is} + \frac{e^{-is} - 1}{s^2} \right]$$

3. Solve the heat problem $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ on $[0,\infty)$ for $x \ge 0$ and $t \ge 0$, with $u(x,0) = f(x)$ for $x \ge 0$, and $u(0,t) = 0$ for all $t \ge 0$. $\leftarrow$ $\hat{u}(w,t) = \tilde{\mathcal{F}}_s[u(x,t)](w)$

4. Solve the heat problem $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ on $(-\infty,\infty)$ for $-\infty \le x \le \infty$ and $t \ge 0$, with $u(x,0) = f(x)$ for all x. $\leftarrow$ $\hat{u}(s,t) = \tilde{\mathcal{F}}[u(x,t)](s)$

3. $$\frac{\partial \hat{u}}{\partial t}(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty \underbrace{\frac{\partial}{\partial t} u(x,t)}_{\frac{\partial^2 u}{\partial x^2}} \sin wx \, dx = \tilde{\mathcal{F}}_s\left[\frac{\partial^2 u}{\partial x^2}\right]$$

$$= -w^2 \tilde{\mathcal{F}}_s[u] + \sqrt{\frac{2}{\pi}} \underbrace{u(0,t)}_{=0}$$

[$\tilde{\mathcal{F}}_c$ for insulated at $x=0$]

$$\boxed{\frac{\partial \hat{u}}{\partial t} = -w^2 \hat{u}}$$

$$\hat{u}(w,t) = C(w) e^{-w^2 t}$$ $C(w)$ real valued

$$\hat{u}(w,0) = \underline{\underline{C(w)}} = \tilde{\mathcal{F}}_s[\underbrace{u(x,0)}_{f(x)}] = \tilde{\mathcal{F}}_s[f]$$

Sol^n: $$u = \tilde{\mathcal{F}}_s[\tilde{\mathcal{F}}_s[u]] = \sqrt{\frac{2}{\pi}} \int_0^\infty [\underbrace{\text{Prob } 2}_{\tilde{\mathcal{F}}_s[f](w)}] e^{-w^2 t} \sin wx \, dw$$

4. $\hat{u}(s,t) = C(s) e^{-s^2 t}$ where $C(s) = \hat{f}(s)$.

$$u(x,t) = \tilde{\mathcal{F}}^{-1}[\hat{f}(s) e^{-s^2 t}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \hat{f}(s) e^{-s^2 t} e^{isx} \, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty \underbrace{\frac{1}{\sqrt{2\pi}} \left(\frac{is e^{-is} + e^{-is} - 1}{s^2} \right)}_{\text{real even, imag odd}} \underbrace{e^{-s^2 t}}_{\text{even}} \underbrace{e^{isx}}_{\cos sx + i \sin sx} \, ds$$ = real fcn

5. Prove Weyl's Criterion, that given a sequence of angles θ_n, the points $e^{i\theta_n}$ are equidistributed on the unit circle if and only if

$$\frac{1}{N}\sum_{n=1}^{N} e^{im\theta_n} \to 0, \quad N \to \infty, \quad m = 1, 2, 3, \ldots$$

tends to zero as N tends to infinitey for each positive integer m.
Hints: The limit for a postive integer m implies the limit for negative integer $-m$ because one is the conjugate of the other. The $m = 0$ case is special (and different). Follow the proof of Weyl's Theorem: true for Fourier basis functions implies true for trigonometric polynomials implies true for continuous 2π-periodic functions implies ...

Weyl's lemma: want

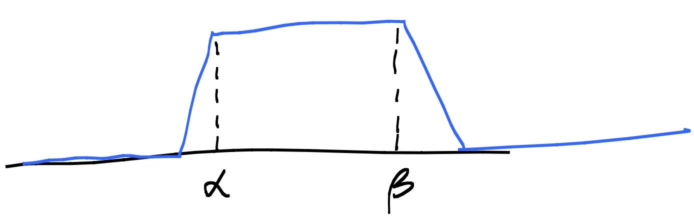
$$\left[\frac{1}{N}\sum_{n=1}^{N} \chi_{[\alpha,\beta]}(e^{i\theta_n}) \to \frac{(\beta-\alpha)}{2\pi}\right]$$

f cont

$$\frac{1}{N}\sum_{n=1}^{N} f(e^{i\theta_n}) \to \frac{1}{2\pi}\int_0^{2\pi} f(\psi)\,d\psi$$

Criterion $\Rightarrow$ lemma for Fourier basis fcns $e^{im\theta}$ for $m = \pm 1, \pm 2, \ldots$

Check: $m=0$ works too. Easy!



etc.