Lesson 35 The Fundamental Theorem of Algebra

Exam 2 due Monday, 4/17
11:00pm in Gradescope

Thm: A polynomial of degree $N \geq 1$ has a root in $\mathbb{C}$.

Polynomials: Complex polynomials $P(z) = a_N z^N + \cdots + a_1 z + a_0$

where a's and z are complex numbers.

EX: $x^2 + 1 = (x-i)(x+i)$

Euclidean algorithm: $P(z) = Q(z)\,G(z) + R(z)$, $\deg R < \deg G$

$$P(z) = Q(z)\,(z - \underset{\substack{\uparrow \\ \text{root} \\ \text{of } P(z)}}{z_0}) + (\text{const})$$

Plug in z_0: $0 = P(z_0) = Q(z_0) \cdot 0 + \underbrace{(\text{const.})}_{\text{must be zero}}$

See $P(z) = Q(z)(z - z_0)$, $\deg Q = \deg P - 1$

Repeat until $\deg Q = 1$.

Get $P(z) = a_N (z - z_1)(z - z_2) \cdots (z - z_N)$ $\quad N = \deg P$, $a \neq 0$

(OK if some z_n's repeat.)

Motivation: Dirichlet problem on unit disc.

Did separation of variables on polar version of $\Delta u = 0$.

Found sol^{n}s $u(r,\theta) = R(r)\Theta(\theta) = \begin{cases} 1 \\ r^n \cos n\theta \\ r^n \sin n\theta \end{cases}$ harmonic!

Try $u(r,\theta) = a_0 + \sum_{n=1}^{\infty} (a_n r^n \cos n\theta + b_n r^n \sin n\theta)$

want

B.C. $u(1,\theta) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) = f(\theta)$

a's, b's are just Fourier coeff for $f(\theta)$.

Interesting thing: $z = re^{i\theta}$. $z^n = (re^{i\theta})^n = r^n e^{in\theta}$

$$= (r^n \cos n\theta) + i(r^n \sin n\theta)$$

Aha! $\operatorname{Re} z^n$ and $\operatorname{Im} z^n$ are harmonic!

Consequently, complex polys satisfy the ave property.

Complex analysis Complex polys are complex diff'ble:

Def^n: $f(z)$ is complex diff'ble at $a \in \mathbb{C}$ if

$\lim_{z \to a} \frac{f(z) - f(a)}{z - a}$ exists. Call limit $f'(a)$.

EX: Constant f'cns ✓

z ✓

$f(z) = z^2$

$$DQ = \frac{z^2 - a^2}{z - a} = \frac{(z-a)(z+a)}{z-a}$$

$$= z + a \xrightarrow[z \to a]{} a + a = 2a$$

Fact: Freshman calculus facts hold true with same proof.

EX: If and f and g are $\mathbb{C}$-diff'ble at a, then

so is $h(z) = f(z)g(z)$ and $h'(a) = f'(a)g(a) + f(a)g'(a)$.

$$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2} \quad \text{when } g(a) \neq 0.$$

<u>Def^n</u>: $\lim_{z \to a} f(z) = L$ means, given $\varepsilon > 0$, $\exists\ \delta > 0$ such that $|f(z) - L| < \varepsilon$ when $|z - a| < \delta$ and $z \neq a$. [Same as $\mathbb{R}^2 \to \mathbb{R}^2$ concept.]

$f(z)$ is in $D_\varepsilon(L)$ when $z \in D_\delta(a) - \{a\}$.

<u>Key fact</u>: Complex rational fcn $R(z) = \frac{P(z)}{Q(z)}$ is $\mathbb{C}$-diff where denom poly Q is $\neq 0$.

<u>Will show</u>: Real and imag parts of $\mathbb{C}$-rational fcns are harmonic!

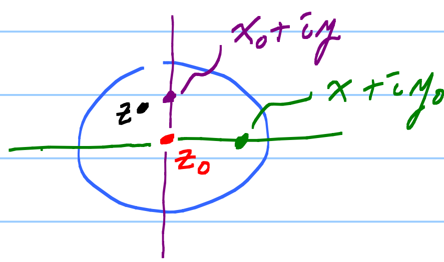
<u>Cauchy-Riemann eqns</u> (MA 425) $\quad w = f(z)$

$$f(z) = f(x + iy) = u(x,y) + i\,v(x,y)$$

<u>Fact</u>: If f is $\mathbb{C}$-diff'ble at $z_0 = x_0 + iy_0$, then

C-R eqns
$$\begin{cases} \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} \\[2ex] \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x} \end{cases} \quad \text{when } (x,y) = (x_0, y_0)$$

<u>Pf</u>: Easy



Green and purple slides go to <u>same</u> limit $f'(z_0)$.

Horizontal case $\quad DQ = \dfrac{f(z) - f(z_0)}{z - z_0}$

$$= \frac{[u(x,y_0) + i v(x,y_0)] - [u(x_0,y_0) + i v(x_0,y_0)]}{(x + i y_0) - (x_0 + i y_0)}$$

$$= \underbrace{\frac{u(x,y_0) - u(x_0,y_0)}{x - x_0}}_{\text{exists as } x \to x_0} + i \underbrace{\frac{v(x,y_0) - v(x_0,y_0)}{x - x_0}}_{\text{same}}$$

$$\rightarrow \quad \boxed{f'(z_0) = \frac{\partial u}{\partial x}(x_0,y_0) + i \frac{\partial v}{\partial x}(x_0,y_0)} \quad \text{Red box \#1}$$

Vertical case : $\quad DQ = \dfrac{\cdots\cdots}{(x_0 + i y) - (x_0 + i y_0)}$

$$= \frac{u(x_0,y) - u(x_0,y_0)}{i(y - y_0)} + \frac{i\,(v(x_0,y) - v(x_0,y_0))}{i(y - y_0)}$$

$$\frac{1}{i} = -i$$

$$\boxed{f'(z_0) = \frac{\partial v}{\partial y}(x_0,y_0) - i \frac{\partial u}{\partial y}(x_0,y_0)} \quad \text{Red box \#2}$$

Equate red boxes to get C-R eqns.

Thm: f $\mathbb{C}$-diff'ble at $z_0 \implies$ first partials of u and v exist at (x_0,y_0) and satisfy C-R eqns.

Cor: Real and imag parts of complex rational fcns are harmonic!

Pf: Step one: $R(z)$ can be $\mathbb{C}$-diff'ed infinetly many times.

$$R(x+iy) = u(x,y) + i\,v(x,y)$$

$$R'(z) = \underbrace{\frac{\partial u}{\partial x}}_{U} + i\,\underbrace{\frac{\partial v}{\partial x}}_{V} \quad \left(= \frac{\partial v}{\partial y} - i\,\frac{\partial u}{\partial y} = U + iV \right)$$

Can do C-R equs again on R''

$$u_{xx} = \frac{\partial}{\partial x}\Big(\underbrace{\frac{\partial u}{\partial x}}_{U}\Big) = \frac{\partial}{\partial y}\Big(\underbrace{\frac{\partial v}{\partial x}}_{V}\Big) = \frac{\partial}{\partial x}\Big(\underbrace{\frac{\partial v}{\partial y}}_{U}\Big) = \frac{\partial}{\partial y}\Big(\underbrace{-\frac{\partial u}{\partial y}}_{V}\Big) = -u_{yy}$$

$\Delta u = 0$!

$$\frac{\partial}{\partial y}\Big(\underbrace{\frac{\partial u}{\partial x}}_{U}\Big) = -\frac{\partial}{\partial x}\Big(\underbrace{\frac{\partial v}{\partial x}}_{V}\Big) = \frac{\partial}{\partial y}\Big(\underbrace{\frac{\partial v}{\partial y}}_{U}\Big) = -\frac{\partial}{\partial x}\Big(\underbrace{-\frac{\partial u}{\partial y}}_{V}\Big)$$

$$-v_{xx} = v_{yy} \qquad \Delta v = 0!$$

u v are harmonic!