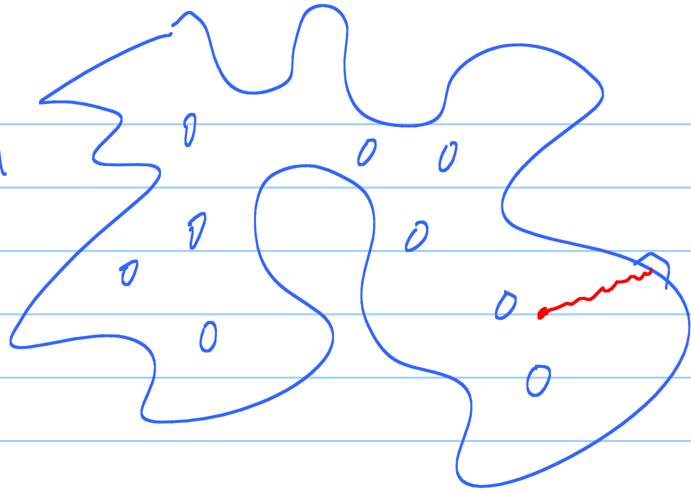
Lesson 37 Research in Fourier Analysis / Harmonic Analysis

Exam 2 papers returned in Brightspace this afternoon. Stats by email later.

Math blob



Heavy theorem from recent past: Lennart Carlesson: $f \in L^2[-\pi,\pi]$ (Lebesgue sense). Know Fourier S. $S_N(x) = \sum_{-N}^{N} c_n e^{inx} \xrightarrow{L^2} f$.

Also know $S_N(x_0)$ might <u>not</u> converge at certain pts x_0, even if f is continuous.

<u>Thm</u>: $S_N(x) \longrightarrow f(x)$ pointwise "almost everywhere", meaning there is a set $S \subset [-\pi,\pi]$ of <u>measure zero</u> such that $S_N(x) \to f(x)$ for $x \in [-\pi,\pi] - S$.

<u>Pf</u>: Fiendishly clever, difficult, long proof.

<u>My story</u>. Solving Dirichlet problem with polynomial data.

On the unit disc: $\mathcal{P}(x,y)$ is a polynomial function of x, y. Let u be sol^n to D. Prob on $D_1(0)$.

<u>Claim</u>: u is a poly!

Why: $\varphi(x,y) = P(x,y) = P\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) = \sum_{n,m \le N} c_{nm} z^n \bar{z}^m$

poly with real coeff

Hmmm. When $|z|=1$, $z^n \bar{z}^n = (\underbrace{z \cdot \bar{z}}_{|z|^2=1})^n = 1$

Know: $\operatorname{Re} z^n$, $\operatorname{Im} z^n$ are harmonic.

Pick a term: $c_{nm} z^n \bar{z}^m$

Case $n=m$: $c_{nm}(z\bar{z})^n = c_{nm}$ when $|z|=1$

extends harmonically to $D_1(0)$

Case $n<m$: $c_{nm} z^n \bar{z}^m = c_{nm}(z^n \bar{z}^n)\bar{z}^{m-n}$

$= c_{nm} \bar{z}^{m-n}$ when $|z|=1$

$=$ Conjugate of $\bar{c}_{nm} z^{m-n}$, so real and imag parts are harmonic. Use them to extend harmonically to $D_1(0)$.

Case $n>m$: $c_{nm} z^n \bar{z}^m = c_{nm} \underbrace{z^m \bar{z}^m}\, z^{n-m}$

$= c_{nm} z^{n-m}$ when $|z|=1$. Use it to extend to $D_1(0)$.

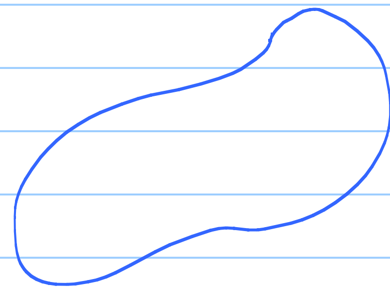
Get $\varphi(x,y) = \underbrace{\sum A_n z^n + \sum B_n \bar{z}^n}_{\text{These extend to } D_1(0)}$ when $|z|=1$.

$\varphi(x,y) = u(x,y) + i\,v(x,y)$ where u, v are harmonic polys! φ real valued $\Rightarrow$ $v = 0$ on $C_1(0)$.

Uniqueness of sol^n to D. Prob $\Rightarrow$ $v \equiv 0$ on $D_1(0)$.

(Max princ)

How special is this?



Ω

Ques: Is the disc the only domain such that (poly data on bndry) yields poly sol^n's to D. Prob.?

Ans: No! Ellipses do too.

Why: Ellipse: $E = \left\{ (x,y) : \frac{x^2}{a^2} + \frac{y^2}{b^2} < 1 \right\}$

(Circle when $a = b$)

Polynomal defining fcn: $\Omega = \{ (x,y) : \underbrace{p(x,y)}_{\text{poly}} < 0 \}$

Let $r(x,y) = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1$

be the quadratic poly defining fcn for E.

Fisher map $\tilde{\Delta}$: Poly $p(x,y) \longmapsto \Delta(rp)$, also a poly.

Hmmm. $p \mapsto rp$ $\leftarrow$ increases degree of poly by 2.

Δp makes degree go down by 2.

So $\tilde{\Delta} : \mathcal{P}_N \longrightarrow \mathcal{P}_N$, $\mathcal{P}_N$ = Polys deg N or less.

Note: $\tilde{\Delta}$ is linear.

$\mathcal{P}_N$ is a finite dim$^{\ell}$ vector space.

Claim: $\tilde{\Delta} : \mathcal{P}_N \to \mathcal{P}_N$ is one-to-one!

Why: Need to show $\tilde{\Delta} p \equiv 0 \implies p \equiv 0$.

$$\Delta(rp) \equiv 0$$

This means rp is harmonic on $D_1(0)$.

and $r = 0$ on boundary of E

implies $rp = 0$ on bE.

Uniquenes, D. Prob $\implies$ $rp \equiv 0$ on E.

So $p \equiv 0$ ✓

Linear alg fact: $\tilde{\Delta} : \mathcal{P}_N \to \mathcal{P}_N$, $\mathcal{P}_N$ finite dim$^{\ell}$, $\tilde{\Delta}$ linear

$\tilde{\Delta}$ one-to-one $\implies$ $\tilde{\Delta}$ onto.

Pf of big fact: Suppose p is a poly.

Let $Q = \Delta p \leftarrow$ poly of deg N.

Know $\exists$ poly q of deg N or less such

$$\tilde{\Delta} q = Q$$

$$\Delta(rq) = \Delta p$$

Aha! $\Delta(\underbrace{p - rq}_{\text{harmonic!}}) = 0$

and $p - rq = p$ on bE because $r = 0$ there.

$p - rq$ is a poly sol^n to D. Prob with bndry data p.

Khavinson-Schapiro conjecture: Discs and ellipses are only domains with the poly $\to$ poly property.

Still open.