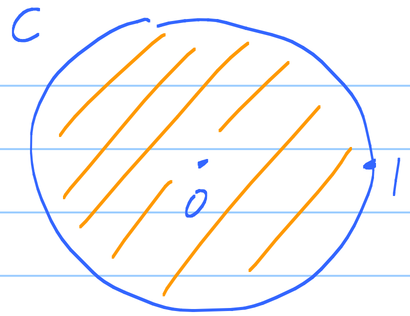
Lesson 38 Vibrating circular membrane (drum) and Bessel functions

C



$$\frac{\partial^2 u}{\partial t^2} = \Delta u$$

$u = 0$ on C

$$\begin{cases} u(r,\theta,0) = f(r,\theta) \\ \frac{\partial u}{\partial t}(r,\theta,0) = g(r,\theta) \end{cases}$$

Sol^n's to Exam 2 posted on home page.

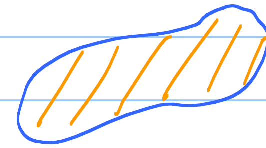
Try $u(r,\theta,t) = R(r)\,\Theta(\theta)\,T(t)$

Khavinson, Schapiro, Ebenfelt, Bell:

(Rational fcns data) $\longrightarrow$ Rational sol^{n}s to D. prob.

Then Ω must be a disc!

Khavinson–Schapiro conj:



Ω

(Poly bndry data) $\longrightarrow$ poly sol^n's to D. prob.

Then Ω must be a disc or an ellipse. Open problem.

Separation of variables: $$\frac{T''}{T} = \boxed{\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} = \lambda}$$

$T'' - \lambda T = 0$. easy.

Multiply red box by r^2: $$r^2\frac{R''}{R} + r\frac{R'}{R} - \lambda r^2 = -\frac{\Theta''}{\Theta} = \mu$$

Θ problem: $\Theta'' + \mu\Theta = 0$ with periodic bndry cond $\begin{cases} \Theta(0) = \Theta(2\pi) \\ \Theta'(0) = \Theta'(2\pi) \end{cases}$

$\mu = n^2$. Θ's Fourier basis fcns.

R-prob $$r^2\frac{R''}{R} + r\frac{R'}{R} - \lambda r^2 = n^2$$

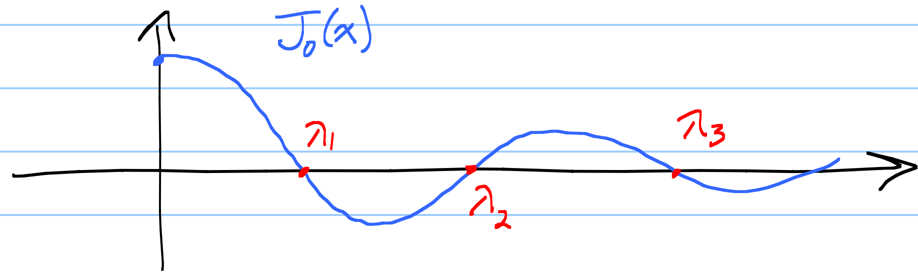
Mult by $R(r)$: $$r^2 R'' + r R' - (\lambda r^2 + n^2) R = 0$$

$$R(1) = 0$$

$$R(0) \text{ finite!}$$

Bessel's eqn. of order n (after scaling r-variable).

Bessel fcn $J_0(x)$ of first kind of order 0.



$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \cdots$$

Find this formula for J_0: Plug power series $\sum_0^\infty c_n x^n$ into ODE and get c_n's recursively.

Radius of conv is infinite. Power series converges fast, faster than e^{-x}, $\cos x$, $\sin x$!

Get $R_{mn}(r) = J_n(\beta_m r)$ where β_m are positive zeroes of J_m.

<u>Sturm–Liouville</u>: R_{mn} are $\perp$ on $[0,1]$ with weight fcn r: $\langle u, v \rangle = \int_0^1 u(r) v(r)\, r\, dr$.

Whole theory like Fourier series works!

Sol^n $$u(r,\theta,t) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} A_{nm} J_n(\beta_m r)\cos n\theta \cos \beta_m t$$
$$+ B_{nm} J_n(\beta_m r)\sin n\theta \cos \beta_m t$$
$$+ C_{nm} J_n(\beta_m r)\cos n\theta \sin \beta_m t$$
$$+ D_{nm} J_n(\beta_m r)\sin n\theta \sin \beta_m t$$

Finally, get coeff from I.C. and $\perp$.

Remark: We did the D. prob for disc in polar coord.

$\Delta u = 0$. Separation var: $R(r)$ problem:
Euler eqn:
$$r^2R'' + rR' + \lambda R = 0$$

Heat eqn or wave eqn lead to Bessel eqns in R.

$\mathbb{R}^3$ versions: Heat eqn on a ball
Wave eqn on a ball

$R(r)$ problem: Legendre equation

Get orthogonal fcns: Legendre polynomials.

Old final and practice problems on home page.

Monday: Go over Exam 2

Remark: Complex Fourier transform to solve the heat eqn.

$$\boxed{\frac{\partial \hat{u}}{\partial t} = -s^2 \hat{u}}$$

$$\hat{u}(s,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t)\, e^{-isx}\, dx$$

is likely a complex valued fcn.

Write $\hat{u}(s,t) = A(s,t) + i B(s,t)$

Plug into ODE. Equate real and imag parts

$$\begin{cases} \dfrac{\partial A}{\partial t} = -s^2 A & A = a(s) e^{-s^2 t} \\[2ex] \dfrac{\partial B}{\partial t} = -s^2 B & B = b(s) e^{-s^2 t} \end{cases}$$

So $\hat{u}(s,t) = a(s) e^{-s^2 t} + i\, b(s) e^{-s^2 t}$

$$= \big(\underbrace{a(s) + i\, b(s)}_{C(s)}\big)\, e^{-s^2 t}$$

Preferred method.

$$\frac{\partial \hat{u}}{\partial t} = -s^2 \hat{u}$$

$$e^{-s^2 t} \cdot \left[\frac{\partial \hat{u}}{\partial t} + s^2 \hat{u} = 0 \right]$$

$$\frac{\partial}{\partial t}\left[e^{-s^2 t}\, \hat{u} \right] = 0$$

So $e^{-s^2 t}\, \hat{u}$ = const which depends on s.

OK to be complex.