Review 3

Office hours: M, W 2:00-3:00pm

The Final Exam is Wednesday, May 3, 7:00-9:00 pm in SCHM 313 = REC 313

The final exam will be closed book, no computers, calculators, phones, or smart watches, and closed notes. You will be allowed to bring crib sheets consisting of two sheets of regular sized paper hand-written only on both sides. As mentioned above, this Formula sheet will be the last page of the exam.

on home page.

Show that the Fourier inversion formula implies the real Fourier integral identity on Schwartz f'cns.

Assume $f \in \mathcal{S}$ real valued.

$$\underbrace{f(x)}_{\text{real valued}} = \mathcal{F}^{-1}[\underbrace{\mathcal{F} f}_{\hat{f}(s)}] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(s)\, e^{+isx}\, ds$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) \underbrace{e^{-ist}}_{\cos st - i \sin st} dt \right] \underbrace{e^{isx}}_{\cos sx + i \sin sx}\, ds$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} \left[\underbrace{\frac{1}{\pi}\int_{-\infty}^{\infty} f(t) \cos st\, dt}_{A(s) \text{ even}} - i \underbrace{\frac{1}{\pi}\int_{-\infty}^{\infty} f(t) \sin st\, dt}_{B(s) \text{ odd}} \right] (\cos sx + i \sin sx)\, ds$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} [\underset{\text{even}}{A(s)} - i \underset{\text{odd}}{B(s)}] (\underset{\text{even}}{\cos sx} + i \underset{\text{odd}}{\sin sx})\, ds$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} A(s) \cos sx + B(s) \sin sx\, ds + i \underbrace{[\text{who cares!}]}_{\text{must } = 0,}$$

$$= \int_0^{\infty} A \cos + B \sin\, ds$$

5. Given that $f(x) = \dfrac{2}{\pi}\displaystyle\int_0^\infty \frac{\cos sx}{1+s^2}\,ds$, what is the Fourier Transform of $f(x)$? Explain.

Hint: Inverse Fourier Transform.

Hmmm. $\underbrace{\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty}\frac{1}{1+s^2}e^{-isx}\,ds}_{\tilde{\mathcal{F}}\left[\frac{1}{1+s^2}\right]} = \underbrace{\dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty}\underbrace{\frac{\cos sx}{1+s^2}}_{\text{even}}\,ds}_{\frac{2\pi}{\sqrt{2\pi}\,\pi}\int_0^\infty \frac{\cos sx}{1+s^2}ds} - \underbrace{\dfrac{i}{\sqrt{2\pi}}\displaystyle\int_{-\infty}^{\infty}\underbrace{\frac{\sin sx}{1+s^2}}_{\text{odd}}\,ds}_{=\lim \int_{-A}^{A} = 0}$

$$= \sqrt{\frac{\pi}{2}}\,f(x)$$

$$f(x) = \sqrt{\frac{2}{\pi}}\,\tilde{\mathcal{F}}\left[\frac{1}{1+s^2}\right]$$

Hit with $\tilde{\mathcal{F}}$:

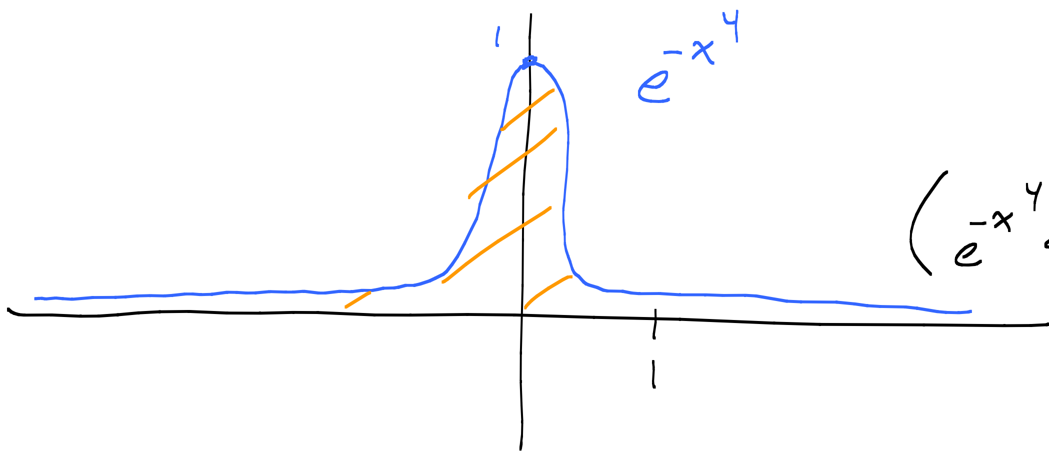
$$\tilde{\mathcal{F}}[f] = \sqrt{\frac{2}{\pi}}\,\tilde{\mathcal{F}}\left[\tilde{\mathcal{F}}\left[\frac{1}{1+s^2}\right]\right] = \sqrt{\frac{2}{\pi}}\left(\frac{1}{1+\underbrace{(-s)^2}_{s^2}}\right)$$

Claim: $\tilde{\mathcal{F}}[\tilde{\mathcal{F}}g] = g(-x)$

$$\tilde{\mathcal{F}}[\hat{g}(s)] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{g}(s)\underbrace{e^{-isx}}_{e^{i(-x)s}}\,ds$$

$$= \tilde{\mathcal{F}}^{-1}[\hat{g}](-x) = g(-x) \quad \checkmark$$

1. Explain how to turn e^{-x^4} into a "bump function."



$\left(e^{-x^4} < e^{-x}, \ x > 1 \right)$

Let $c = \int_{-\infty}^{\infty} e^{-x^4}\, dx < \infty$.

Let $\varphi(x) = \frac{1}{c} e^{-x^4}$.

Bump fcn $\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi(x/\varepsilon) = \frac{1}{c\varepsilon} e^{-(x/\varepsilon)^4}$

$\left(\widetilde{\mathcal{F}}\left[e^{-x^4}\right] \text{ not itself} \right)$

7. Practice problem 1 for Exam 2 showed that

$$\alpha(w) = \int_{-\infty}^{\infty} g(x) \sin wx \, dx$$

Let $\varepsilon > 0$.

goes to zero as $w \to \infty$ when $g(x)$ is a real valued C^1-smooth function on the whole real line in $L^1(\mathbb{R})$, meaning that $\int_{-\infty}^{\infty} |g(x)| \, dx$ is finite. Show that the same is true for $g \in L^1(\mathbb{R})$ that are merely continuous. What if g is allowed to have finitely many jumps?

Idea. Still know $\exists\, N > 0$ $\left| \int_{|x|>N} g(x) \sin wx \, dx \right| < \frac{\varepsilon}{3}$

Aha! Use Weierstraß approx thm to get poly $p(x)$ with

$$|g(x) - p(x)| < \frac{\varepsilon/3}{2N} \quad \text{on } [-N, N].$$

$$\int_{-N}^{N} g(x) \sin wx \, dx = \underbrace{\int_{-N}^{N} (g(x) - p(x)) \sin wx \, dx}_{\left| \quad \right| \leq \int_{-N}^{N} |g-p| \cdot 1 \, dx < \frac{\varepsilon}{3}} + \underbrace{\int_{-N}^{N} p(x) \sin wx \, dx}_{\text{Finally, integrate by parts. Make } < \frac{\varepsilon}{3} \text{ by taking } w \text{ large enough}}$$

Riemann-Lebesgue lemma: Fourier series coeff go to zero as $n \to \infty$.

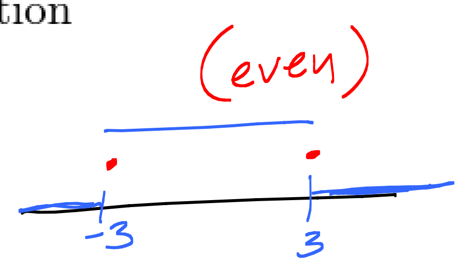
MA 428 version: Bessel's ineq $\Rightarrow$ this for piecewise cont fcns.

Riemann: $f \in L^1[-\pi, \pi] \leftarrow \int_{-\pi}^{\pi} |f| dx < \infty \leftarrow$ Riemann integral

Lebesgue: same ... $\leftarrow$ Lebesgue int.

4. Compute the Fourier transform of the even function

$$f(x) = \begin{cases} 1 & -3 < x < 3 \\ 0 & \text{elsewhere.} \end{cases}$$



What is the Fourier transform of the Fourier transform of f?
Be careful to give the value of the function for *every* real number.

Know $\tilde{\mathcal{F}}[\tilde{\mathcal{F}} f] = f(-x) = f(x)$ at $x \neq \pm 3$.

At jumps, $=$ midpoint of jump of $f(-x) = f(x)$

$$= \frac{1}{2}$$

Fact: f piecewise C^1-smooth with sufficient decay cond at ∞.

$$\tilde{\mathcal{F}}^{-1}[\tilde{\mathcal{F}} f] = \begin{cases} f(x) & x \text{ smooth near } x \\ \text{midpt. jumps} & x \text{ jump pt} \end{cases}$$

$$\tilde{\mathcal{F}} f = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \left[\cos sx - i \sin sx\right] dx \rightarrow 0 \text{ as } s \to \infty$$

$$= \frac{1}{\sqrt{2\pi}} \left[\underbrace{\int_{-\infty}^{\infty} f(x) \cos sx \, dx}_{\to 0 \text{ as } s \to \infty} - i \underbrace{\int_{-\infty}^{\infty} f(x) \sin sx \, dx}_{\to 0 \ s \to \infty} \right]$$

when f is cont on $\mathbb{R}$ and $\int_{-\infty}^{\infty} |f| \, dx < \infty$.