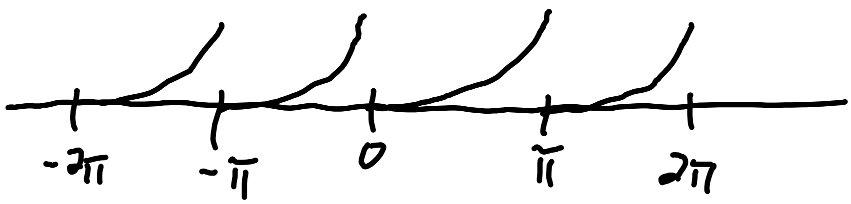
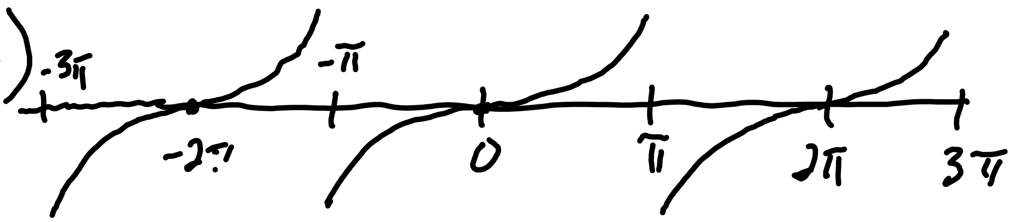
MA 428 homework 4 sol^{n}s

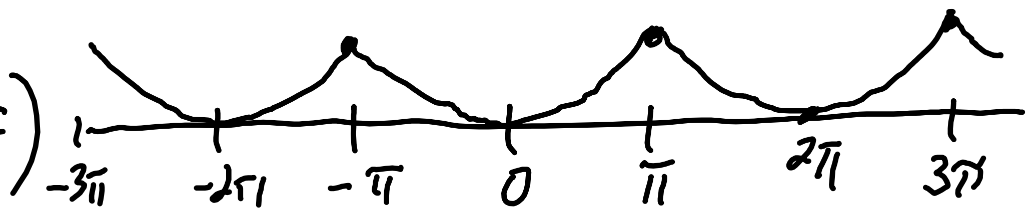
1. a)



b)



c)



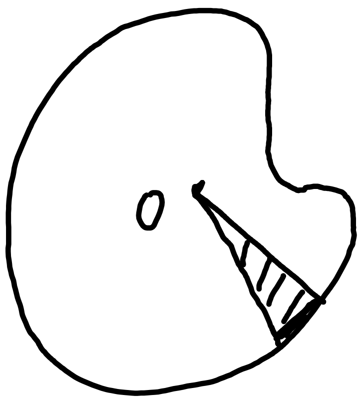
2) Parseval's:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\,dx = 2a_0^2 + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

$$\frac{1}{\pi}\int_{-\pi}^{\pi} \left(x^2\right)^2 dx = 2\left(\frac{\pi^2}{3}\right)^2 + 16\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x^4\,dx = \frac{2}{\pi}\left(\frac{1}{5}\cdot\pi^5 - 0\right) = \frac{2\pi^4}{5}$$

So $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{1}{16}\left[\frac{2\pi^4}{5} - \frac{2\pi^4}{9}\right] = \pi^4 \frac{8}{16\cdot 45} = \frac{\pi^4}{90}$

3. 

$dA = \frac{1}{2} r (r\,d\theta) = \frac{1}{2} r^2 \, d\theta$

$$A = \frac{1}{2}\int_0^{2\pi} f(\theta)^2 \, d\theta$$

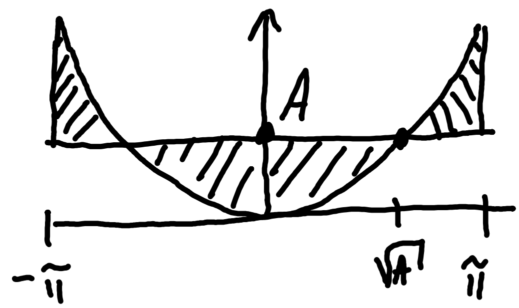
$$= \pi \sum_{n=-\infty}^{\infty} |c_n|^2 \leftarrow \quad c_n = \hat{f}(n)$$

or $$= \frac{\pi}{2}\left[2a_0^2 + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)\right]$$

via <u>Parseval's</u> identity.

4. a) $|f(0)-A|$ is minimum when

$A = f(0) = 0.$

b) 

$a(A) = \text{area}$

$= \int_{-\pi}^{\pi} |x^2 - A|\,dx$

Since $a(A) > a(\pi^2)$ if $A > \pi^2$ and

$a(A) > a(0)$ if $A < 0$, we may

restrict A to be $0 \le A \le \pi^2$.

$$a(A) = 2\left(\int_0^{\sqrt{A}} (A - x^2)\,dx + \int_{\sqrt{A}}^{\pi} (x^2 - A)\,dx \right)$$

$$= 2\left(\left[Ax - \frac{x^3}{3} \right]_0^{\sqrt{A}} + \left[\frac{x^3}{3} - Ax \right]_{\sqrt{A}}^{\pi} \right)$$

$$= 2\left(A^{3/2} - \frac{1}{3}A^{3/2} + \frac{\pi^3}{3} - A\pi - \frac{A^{3/2}}{3} + A^{3/2} \right)$$

$$= 2\left(\frac{4}{3}A^{3/2} - A\pi + \pi^3/3 \right)$$

$$a'(A) = 4A^{1/2} - 2\pi = 0 \text{ at critical point.}$$

$A = \frac{\pi^2}{4}$ is a critical point.

$$a(0) = 2\int_0^{\pi} x^2\,dx = \frac{2\pi^3}{3}$$

$$a(\pi^2) = 2\int_0^{\pi} (\pi^2 - x^2)\,dx = 2\left(\pi^3 - \frac{\pi^3}{3}\right) = \frac{4\pi^3}{3}$$

$$a\left(\frac{\pi^2}{4}\right) = 2\left(\frac{4}{3}\left(\frac{\pi^2}{4}\right)^{3/2} - \left(\frac{\pi^2}{4}\right)\pi + \frac{\pi^3}{3}\right)$$

$$= \frac{\pi^3}{3} - \frac{\pi^3}{2} + \frac{2\pi^3}{3}$$

$$= \frac{\pi^3}{2} \leftarrow \text{min.}$$

The min occurs at $A = \frac{\pi^2}{4}$

c) $F_0(x) = c_0$ is the trig poly that gives the best L^2 approximation to $f(x)=x^2$ on $[-\pi,\pi]$ among all trig polys of degree zero. So $A=c_0=\frac{\pi^2}{3}$

d) Let $M(A)=\text{Max}\{|x^2-A|: -\pi\le x\le A\}$.

Since $M(A)>M(\pi^2)$ when $A>\pi^2$ and $M(A)>M(0)$ when $A<0$, we may restrict A to $0\le A\le \pi^2$.

If $0\le A\le \pi^2$, then

$$M(A)=\text{Max}\{\pi^2-A, A\} \text{ is minimum}$$

at $A=\frac{\pi^2}{2}$.