MA 428 HWK 8 solutions

1. Let $\varepsilon > 0$. Since

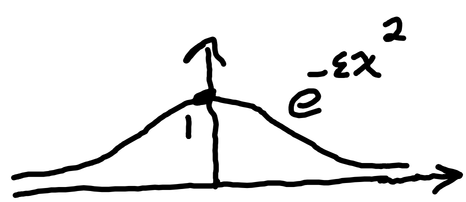
$$\underbrace{\int_{-\infty}^{\infty} |f(x)|\,dx}_{I} = \underbrace{\int_{-N}^{N} |f(x)|\,dx}_{\to I \text{ as } N\to\infty} + \underbrace{\int_{|x|>N} |f(x)|\,dx}_{\text{must} \to 0 \text{ as } N\to\infty},$$

there is a N such that

$$\int_{|x|>N} |f(x)|\,dx < \frac{\varepsilon}{2}.$$

Notice that $\max_{\mathbb{R}} e^{-\varepsilon x^2} = 1$ and

$$\min_{[-N,N]} e^{-\varepsilon x^2} = e^{-\varepsilon N^2}$$



Now $\left| \int_{-\infty}^{\infty} f(x)\,dx - \int_{-\infty}^{\infty} f(x) e^{-\varepsilon x^2}\,dx \right| \leq$

$$\int_{-N}^{N} |f(x)| \underbrace{(1-e^{-\varepsilon x^2})}_{\leq 1-e^{-\varepsilon N^2}} dx + \int_{|x|>N} |f(x)| \underbrace{(1-e^{-\varepsilon x^2})}_{\leq 1} dx$$

$$\leq \underbrace{(1-e^{-\varepsilon N^2})}_{\substack{\to 1-1=0 \\ \text{as } \varepsilon \to 0}} \int_{-N}^{N} |f(x)|\, dx + \underbrace{\int_{|x|>N} |f(x)|\, dx}_{< \frac{E}{2}}$$

The first term can be made less than $\frac{E}{2}$ by choosing $\varepsilon > 0$ sufficiently small. So the integrals differ by less than E and we have proved the limit.

2. $$\hat{f}(s) = \frac{1}{2\pi}\left(\int_1^2 \underbrace{(x-1)}_{u} \underbrace{e^{-isx} dx}_{dv} + \int_2^3 \underbrace{(3-x)}_{u} \underbrace{e^{-isx} dx}_{dv} \right)$$

$$= \frac{1}{2\pi}\left[(x-1)\left(-\frac{1}{is} e^{-isx}\right)\Big|_1^2 - \int_1^2 -\frac{1}{is} e^{-isx} dx \quad + \right.$$

$$\left. (3-x)\left(-\frac{1}{is} e^{-isx}\right)\Big|_2^3 - \int_2^3 -\frac{1}{is} e^{-isx} (-dx) \right]$$

$$= \frac{1}{2\pi}\left[-\frac{1}{is} e^{-i2s} - \frac{1}{(is)^2} e^{-isx} \Big|_1^2 \quad + \right.$$

$$\left. -\left(-\frac{1}{is} e^{-i2s}\right) + \frac{1}{(is)^2} e^{-isx} \Big|_2^3 \right]$$

$$= \frac{1}{2\pi}\left[\frac{1}{s^2}\left(e^{-i2s} - e^{-is}\right) - \frac{1}{s^2}\left(e^{-i3s} - e^{-i2s}\right) \right]$$

$$\hat{f}(s) = \frac{1}{2\pi}\cdot\frac{1}{s^2}\cdot\left(2e^{-i2s} - e^{-i3s} - e^{-is}\right)$$

Next, $$\hat{f}(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\left[\cos sx - i\sin sx\right]dx$$

$$= \frac{1}{2\pi\sqrt{\frac{2}{\pi}}}\left(\tilde{\mathcal{F}}_c f - i\,\tilde{\mathcal{F}}_s f\right)$$

can replace $-\infty$ by zero here because $f \equiv 0$ for $x < 0$.

So $$\tilde{\mathcal{F}}_c f = 2\sqrt{2\pi}\,\mathrm{Re}\,\hat{f}(s)$$

$$= \frac{2}{\sqrt{2\pi}}\,\frac{1}{s^2}\left(2\cos 2s - \cos 3s - \cos s\right)$$

and $$\tilde{\mathcal{F}}_s f = -2\sqrt{2\pi}\cdot\frac{1}{2\pi}\cdot\frac{1}{s^2}\left(-2\sin 2s + \sin 3s + \sin s\right)$$

$$= \frac{2}{\sqrt{2\pi}}\,\frac{1}{s^2}\left(2\sin 2s - \sin 3s - \sin s\right)$$

3. This problem calls for using the Fourier cosine transform because

$$\tilde{\mathcal{F}}_c f'' = -s^2 \tilde{\mathcal{F}}_c f - \sqrt{\tfrac{2}{\pi}}\, \underbrace{f'(0)}$$

will be zero at an <u>insulated</u> end.

PDE: $\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$

BC: Temp gradient $=0$ at $x=0$:

$$\frac{\partial u}{\partial x}(0,t) = 0 \quad \text{for all time.}$$

Apply $\tilde{\mathcal{F}}_c$ to PDE in space variable:

Write $\hat{u}(s,t) = \left(\tilde{\mathcal{F}}_c u\right)(s)$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty u(x,t) \cos sx \, dx.$$

$$\tilde{\mathcal{F}}_c \frac{\partial u}{\partial t} = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial u}{\partial t}(x,t) \cos sx \, dx$$

$$= \frac{\partial}{\partial t}\left(\sqrt{\frac{2}{\pi}} \int_0^\infty u(x,t) \cos sx \, dx \right)$$

if u is sufficiently smooth and $\to 0$ at infinity sufficiently fast, both reasonable expectations for this heat problem. So it is reasonable to expect that

$$\frac{\partial \hat{u}}{\partial t} = \sqrt{\frac{2}{\pi}} \int_0^\infty \underbrace{\frac{\partial u}{\partial t}(x,t)}_{= c^2 \frac{\partial^2 u}{\partial x^2}} \cos sx \, dx$$

$$= c^2 \tilde{\mathcal{F}}_c \left(\frac{\partial^2 u}{\partial x^2} \right) =$$

$$= c^2 \left[-s^2 \tilde{\mathcal{F}}_c u - \underbrace{\sqrt{\tfrac{2}{\pi}} \frac{\partial u}{\partial x}(0,t)}_{=0} \right]$$

$$= -c^2 s^2 \hat{u}(s,t).$$

Aha! $\dfrac{\partial \hat{u}}{\partial t} = -c^2 s^2 \hat{u}$ is an easy

problem: $\hat{u} = C(s)\, e^{-c^2 s^2 t}$

Note that $\hat{u}(s,0) = C(s) \cdot 1$

$$= \sqrt{\tfrac{2}{\pi}} \int_0^\infty f(x) \cos sx \, dx$$

$$= \tilde{\mathcal{F}}_c f = \underbrace{\frac{2}{\sqrt{2\pi}} \cdot \frac{1}{s^2} \left(\cos 2s - \cos 3s - \cos s \right)}_{C(s)}$$

So we have determined $C(s)$ and, consequently, $\hat{u}(s,t) = C(s)\, e^{-c^2 s^2 t}$.

Finally, we get u be using the fact that $\tilde{\mathcal{F}}_c \circ \tilde{\mathcal{F}}_c = id$:

$$u(x,t) = \tilde{\mathcal{F}}_c\left[\hat{u}(s,t)\right](x)$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty C(s)\, e^{-c^2 s^2 t} \cos sx \, ds$$

$$= \frac{2}{\pi} \int_0^\infty \frac{\cos 2s - \cos 3s - \cos s}{s^2}\, e^{-c^2 s^2 t} \cos sx \, ds$$

4. Since this problem has no boundary conditions, the full complex Fourier transform is the proper tool, using

$\tilde{\mathcal{F}} f'' = -s^2 \tilde{\mathcal{F}} f$ at the key point.

This time let

$$\hat{u}(s,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} u(x,t) e^{-isx}\, dx$$

and apply $\tilde{\mathcal{F}}$ to the PDE in the space variable to get

$$\frac{\partial \hat{u}}{\partial t} = -c^2 \hat{u} \quad \text{and}$$

$$u = C(s)\, e^{-c^2 s^2 t}.$$ But this

time $\quad \hat{u}(s,0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)\, e^{-isx}\, dx$

$$= \hat{f}(s).$$

So $C(s) = \hat{f}(s)$. Finally, when we "undo" the Fourier transform by taking the inverse Fourier transform in the space variable s

[Note: $\check{g}(x) = \int_{-\infty}^{\infty} g(s)\, e^{isx}\, dx$]

nothing here

we get

$$u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2e^{-i2s} - e^{-i3s} - e^{-is}}{s^2}\, e^{-c^2 s^2 t}\, e^{isx}\, ds$$

One could further manipulate this answer by noting that the solution is real valued, and so $u(x,t) =$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \left[\text{Real part of integrand}\right] ds.$$

I haven't done much of this in class, but once you have an explicit integral for the solution like this, it is not hard to prove that it really does solve the problem.