HWK 9 sol^{n}s

1. Let $c = \int_{-\infty}^{\infty} e^{-x^4}\,dx$ and $\varphi(x) = \frac{1}{c} e^{-x^4}$. Then $\int_{-\infty}^{\infty} \varphi(x)\,dx = 1$.

$$\varphi_\varepsilon(x) = \frac{1}{\varepsilon}\varphi\left(\frac{x}{\varepsilon}\right) = \frac{1}{\varepsilon}\cdot\frac{1}{c}\cdot e^{-(x/\varepsilon)^4}$$ is a "good kernel."

2. Notice that $\tilde{\mathcal{F}}^{-1}[g](x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(s)\,e^{isx}\,ds =$

$$\tilde{\mathcal{F}}[g](-x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(s)\,e^{-is(-x)}\,ds.$$ So $\tilde{\mathcal{F}}[g](x) = \tilde{\mathcal{F}}^{-1}[g](-x)$.

$$f(x) = \frac{2}{\pi}\int_0^{\infty} \frac{\cos sx \leftarrow}{\underbrace{s^2+1}_{\text{even}} \leftarrow}\text{even}\,ds = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\cos sx}{s^2+1}\,ds$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cos sx}{s^2+1} - i\,\underbrace{\frac{\sin sx \leftarrow \text{odd}}{s^2+1 \leftarrow \text{even}}}_{\text{odd}}\,ds$$

$$= \frac{1}{\pi}\sqrt{2\pi}\;\tilde{\mathcal{F}}\left[\frac{1}{s^2+1}\right](x)$$

$$\tilde{\mathcal{F}}[f](x) = \tilde{\mathcal{F}}^{-1}[f](-x) = \frac{1}{\sqrt{2\pi}}\cdot\sqrt{\frac{2}{\pi}}\;\tilde{\mathcal{F}}^{-1}\left[\tilde{\mathcal{F}}\left[\frac{1}{s^2+1}\right]\right](-x)$$

$$= \frac{1}{\pi}\cdot\frac{1}{(-x)^2+1} = \frac{1}{\pi(x^2+1)}$$

4. Let $\varepsilon > 0$. Since $\int_{|x|>N} |g(x)|\,dx \to 0$ as $N \to \infty$, there is an N_0 such that $\left|\int_{|x|>N} g(x)\sin wx\,dx\right| < \frac{\varepsilon}{3}$ for all w.

Let P be a polynomial such that $|g(x) - P(x)| < \frac{\varepsilon/3}{2N_0}$ on $[-N_0, N_0]$. Since P is C^1-smooth, we know there is an M such that $\int_{-N_0}^{N_0} P(x) \sin wx \, dx < \frac{\varepsilon}{3}$ when $w > M$

from HWK 7. Now

$$\left| \int_{-\infty}^{\infty} g(x) \sin wx \, dx \right| = \left| \int_{|x|>N_0} g(x) \sin wx \, dx \right.$$

$$\left. + \int_{-N_0}^{N_0} [g(x) - P(x)] \sin wx + \int_{-N_0}^{N_0} P(x) \sin wx \, dx \right|$$

$$< \underbrace{\frac{\varepsilon}{3} + \frac{\varepsilon/3}{2N_0} \cdot 2N_0 + \frac{\varepsilon}{3}}_{\varepsilon} \quad \text{when } w > M.$$

5. Same argument in prob 4 shows $\int \to 0$ as $|w| \to \infty$, and even when sine is replaced by cosine. So

$$\tilde{\mathcal{F}}[g](w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{-iwx} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} g(x) \cos wx \, dx}_{\downarrow \, 0} - \frac{i}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} g(x) \sin wx \, dx}_{\downarrow \, 0}$$

as $|w| \to \infty$.