Math 428 Exam 2

1. Suppose

$$f(x) = \begin{cases} 0 & x \le 1 \\ x-1 & 1 < x \le 2 \\ 3-x & 2 < x \le 3 \\ 0 & 3 \le x \end{cases}.$$

Solve the heat problem,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

for $x \ge 0$ and $t \ge 0$, with $u(x,0) = f(x)$, and $u(0,t) = 0$ for all $t \ge 0$.

You may leave your answer in the form of an integral of explicit functions (but not an integral of an integral).

2. Suppose that f is continuous on $\mathbb{R}$. Show that f and its Fourier transform $\hat{f}(s) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} f(x)e^{-isx}\,dx$ cannot both be compactly supported unless $f \equiv 0$.

Facts: The support of f is the closure of the set $\{x : f(x) \neq 0\}$. Hence, if the support of f is compact, i.e., closed and bounded, then f is zero outside $(-B, B)$ for some $B > 0$.

Hints: If f is supported in $[-\pi/2, \pi/2]$ and $\hat{f}(s) = 0$ for $|s| > N$, expand f in a Fourier series in the interval $[-\pi, \pi]$ and show that f must be a trigonometric polynomial of degree less than or equal to N. Can a trig poly be zero on an open interval? (Hint: power series centered at a point in the open interval.)

Now solve the general problem by using some L in place of π and following similar ideas.

3. Find a solution $u(x,t)$ to the following vibrating *infinite* string problem using D'Alembert's method.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad -\infty < x < \infty,\ t > 0$$

with $u(x,0) = e^{-x^2}$ and $\frac{\partial u}{\partial t}(x,0) = 0$ for $-\infty < x < \infty$.

4. Suppose that u is continuous on the closed unit disc $\overline{D_1(0)}$ in the complex plane and that

$$(*) \qquad u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(z + (1-|z|)e^{i\theta})\,d\theta$$

for each $z \in D_1(0)$. (This equality means that u is only known to satisfy the averaging property on internally tangent circles centered at z like the one pictured below.) Prove that u must be harmonic in $D_1(0)$.