Math 428

Review problems

Formula sheet on last page

1. Use D'Alembert's method to solve the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

for the *infinite* vibrating string problem with initial position $u(x,0) = e^{-x^2}$ and zero initial velocity. The solution has one "hump" at time zero. Is that true of the solution for all time?

2. Find the complex Fourier transform and the Fourier cosine transform of e^{-4x^2}.

3. Explain how to turn e^{-x^4} into a "bump function."

4. Every function on $[-\pi, \pi]$ can be written as the sum of an even function and an odd function via

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}.$$

Is this decomposition unique? How are the Fourier series for the three functions related?

5. The Legendre polynomials

$$p_n(x) = \frac{d^n}{dx^n}(1 - x^2)^n$$

are orthogonal on $[-1, 1]$. Let

$$A_n = \int_{-1}^{1} p_n^2 \, dx.$$

To expand a function f in terms of Legendre polynomials, $f = \sum_{n=0}^{\infty} c_n p_n$, what must the coefficients c_n be? What is Bessel's inequality in this context? It can be shown that the linear span of the Legendre polynomials of degree N or less is equal to the vector space of polynomials of degree N or less. Explain why this implies that Bessel's inequality must be an equality when f is assumed to be continuous.

6. Solve $\dfrac{\partial^2 u}{\partial x \partial y} = 0$.

7. Let $\quad f(x) = \begin{cases} 1 & 0 < x < \pi/2 \\ 2 & \pi/2 < x < \pi. \end{cases}$, and let $a_0 + \sum_{n=1}^{\infty} a_n \cos nx$ be its Fourier cosine series for f on $[0, \pi]$. Then $a_0 + \sum_{n=1}^{\infty} a_n \cos n\pi/2 =$

8. Compute the coefficients a_n in problem 1. (Don't bother to simplify the formulas you get.)

9. If the Fourier sine Series of $f(x) = x^2$ on $0 < x < \pi$ is $\sum_{n=1}^{\infty} b_n \sin nx$, then the sum of the infinite series $\sum_{n=1}^{\infty} b_n^2 =$

10. Suppose $f(x) = 1 + \cos x + \sum_{n=1}^{\infty} \dfrac{1}{n^2} \sin nx \quad (-\pi < x < \pi)$. Compute $\int_{-\pi}^{\pi} f(x) \cos 3x \, dx$ and $\int_{-\pi}^{\pi} f(x) \sin 3x \, dx$. Is f continuous? Why or why not?

11. Compute the Fourier transform of the odd extension of f to $\mathbb{R}$ if $f(x) = \begin{cases} x & 0 < x < \pi \\ 0 & x > \pi. \end{cases}$

12. Compute the Fourier transform of the even function $f(x) = \begin{cases} 1 & -2 < x < 2 \\ 0 & \text{elsewhere.} \end{cases}$

What is the Fourier transform of the Fourier transform of f? (Be careful to give the value of the function for *every* real number.)

13. Suppose that $u(r, \theta)$ is the harmonic function (that is, $\Delta u = 0$), given in polar coordinates in the unit disk $r < 1$ having boundary function

$$u(1, \theta) = \sin 2\theta + \cos 2\theta.$$

Then, $u\left(\frac{1}{2}, \frac{\pi}{2}\right) =$

Hint: Recall that the functions $r^n \sin n\theta$ and $r^n \cos n\theta$, being the real and imaginary parts of z^n, are harmonic on the unit disc.

14. Given that $f(x) = \dfrac{2}{\pi} \displaystyle\int_0^{\infty} \dfrac{\cos sx}{1 + s^2} \, ds$, what is the Fourier Transform of $f(x)$? Explain.

Hint: Inverse Fourier Transform.

15. Let $g(x) = \sin 3\pi x$. If $u(x,t)$ is the solution to the problem

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2} \quad 0 < x < 2,\ t > 0$$
$$u(x,0) = g(x) \quad 0 < x < 2$$
$$\frac{\partial u}{\partial t}(x,0) = 0 \quad 0 < x < 2$$
$$u(0,t) = u(2,t) = 0 \quad 0 \le t,$$

then $u(\frac{1}{2}, \frac{1}{12}) =$

Hint: Use D'Alembert's solution $u(x,t) = \phi(x - ct) + \psi(x + ct)$

16. Let $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \sin \frac{(2n+1)\pi x}{2}$ and $u(x,t)$ be the solution to

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad 0 \le x \le 2,\ t > 0$$
$$u(0,t) = u(2,t) = 0 \quad t > 0$$
$$u(x,0) = f(x) \quad 0 \le x \le 2$$

Then $u(x,t) =$

A. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} e^{-n^2\pi^2 t/4} \sin \frac{(2n+1)\pi x}{2}$

B. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \cos \frac{n^2\pi^2 t}{4} \sin \frac{(2n+1)\pi x}{2}$

C. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} e^{-(2n+1)^2\pi^2 t/4} \sin \frac{(2n+1)\pi x}{2}$

D. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \cos \frac{(2n^2+1)^2\pi^2 t}{4} \sin \frac{(2n+1)\pi x}{2}$

E. $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} \cos \frac{(2n+1)\pi t}{2} \sin \frac{(2n+1)\pi x}{2}$

Formulas

Fourier sine series for $f(x)$ on $[0, \pi]$ is $\sum_{n=1}^{\infty} B_n \sin nx$ where

$$B_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \; dx$$

Full Fourier series for $f(x)$ on $[-\pi, \pi]$ is $a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$ where

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \; dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \; dx$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \; dx$$

The complex version is $\sum_{n=-\infty}^{\infty} c_n e^{inx}$ where $c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \; dx$

Facts: $c_0 = a_0$, and if $n \geq 1$: $c_n = \frac{a_n - ib_n}{2}$ and $c_{-n} = \frac{a_n + ib_n}{2}$

Parseval's identities (Bessel's inequality with $\leq$ in place of $=$)

$$2a_0^2 + \sum_{n=1}^{\infty} ({a_n}^2 + {b_n}^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx \quad \text{and} \quad \sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2 dx$$

Fourier transform of f is $\hat{f}(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-isx} \; dx$

Inverse Fourier transform of g is $\check{g}(x) = \int_{-\infty}^{\infty} g(s) e^{isx} \; ds$

Fourier cosine transform of f is $(\mathcal{F}_c f)(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos(sx) \; dx$

Fourier sine transform of f is $(\mathcal{F}_s f)(s) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin(sx) \; dx$

The Plancherel formula is $\int_{-\infty}^{\infty} |f(x)|^2 \; dx = 2\pi \int_{-\infty}^{\infty} |\hat{f}(s)|^2 \; ds$