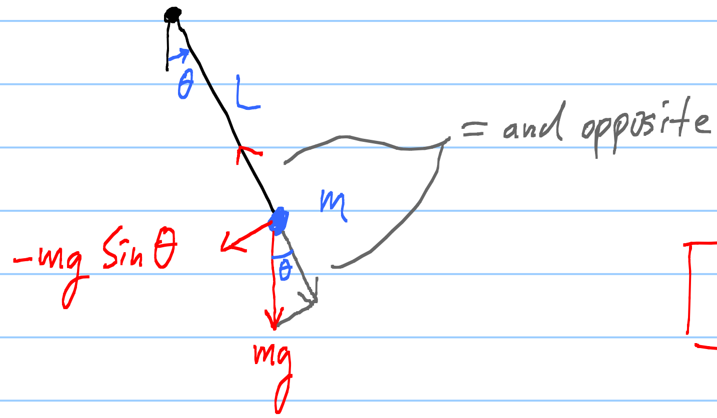
Lesson 13 on 4.5

9,10,11 due tonight, 12,13,14 next Wed.
Exam 1 on Thurs., Oct 1 8-9pm in WTHR 200
covers Lessons 1-14

Damped pendulum: stiff string



$\vec{F} = m\vec{a}$ in tangential dir.

$$-mg\sin\theta \underbrace{-c\dot\theta}_{\text{friction}} = m(L\ddot\theta)$$

$$\boxed{mL\ddot\theta + c\dot\theta + mg\sin\theta = 0}$$

Prob: $\ddot\theta = -\sin\theta - \dot\theta$

Let $\begin{cases} x_1 = \theta(t) \\ x_2 = \dot\theta(t) \end{cases}$ $\quad\begin{cases} x_1' = \dot\theta = x_2 \\ x_2' = \ddot\theta = -\sin\theta - \dot\theta \\ \quad = -\sin x_1 - x_2 \end{cases}$

$$\begin{cases} x_1' = x_2 & \leftarrow f \\ x_2' = -\sin x_1 - x_2 & \leftarrow g \end{cases}$$

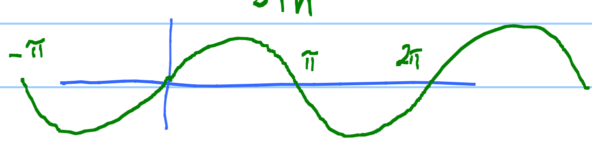
Step 1 Crt. Pts. $\begin{cases} f=0 \\ g=0 \end{cases}$ $\quad x_2 = 0$ ✓

$-\sin x_1 - x_2 = 0$

$\sin x_1 = 0$

↑ happens when $x_1 = n\pi$, $n = 0, \pm1, \pm2, \dots$

sin



$\boxed{(n\pi, 0)}$

Step 2: Linearize at Crt. Pts.

Jacobian matrix $= J = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$ $\qquad f = x_2$

$\qquad g = -\sin x_1 - x_2$

Linearized syst. $\vec{x}' = A\vec{x}$ where

$\mathbb{A} = J\Big|_{(x_0, y_0)}$ ← Crt. pt. new origin

$$\mathbb{A} = \begin{bmatrix} 0 & 1 \\ -\cos x_1 & -1 \end{bmatrix}\Bigg|_{(n\pi, 0)}$$

<u>Case:</u> n even (including $n=0$) $\quad \cos(\text{even})\pi = 1$

$$\mathbb{A} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

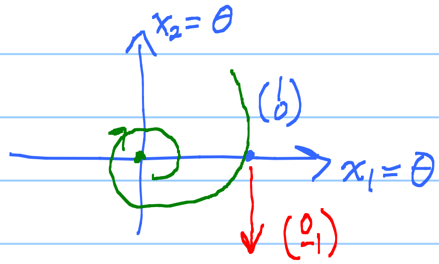
e-vals $(-\lambda)(-1-\lambda)+1 = 0$

$\lambda^2 + \lambda + 1 = 0$

$\lambda = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$ <u>complex</u>

Spiral $\quad$ ↑ $\mathrm{Re}\,\lambda < 0$ <u>in</u>

CW or CCW? $\quad \vec{x}'$ at $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$



$$\begin{cases} \tilde{x} = x - x_0 \\ \tilde{y} = y - y_0 \end{cases}$$

rename x, y

<u>Case:</u> Odd mult π. $\quad \cos(\text{odd})\pi = -1$

$$\begin{array}{c} \mathbb{A} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \\ \uparrow \\ -\cos n\pi \end{array}$$

e-vals $(-\lambda)(-1-\lambda) - 1 = 0$

$\lambda^2 + \lambda - 1 = 0$

$\lambda = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$ opp. sign

Saddle pt., Unstable

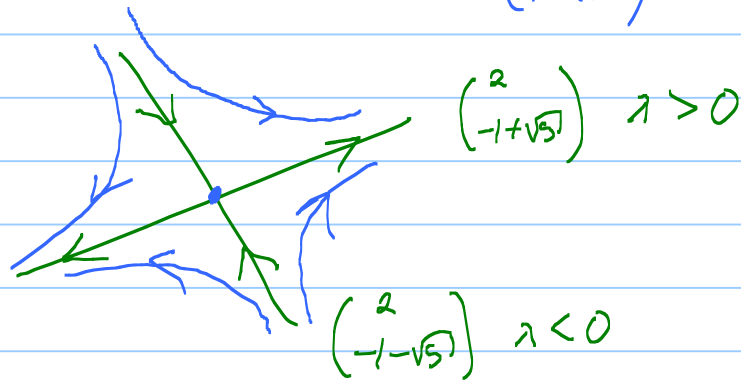
<u>For $\lambda = -\frac{1}{2} - \frac{\sqrt{5}}{2}$:</u> $(\mathbb{A} - \lambda \mathbb{I})\vec{q} = \vec{0}$

$$\left[\begin{array}{cc|c} 0 - \left(-\frac{1}{2} - \frac{\sqrt{5}}{2}\right) & 1 & 0 \\ \text{eeee} & \text{eeee} & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} \frac{1}{2} + \frac{\sqrt{5}}{2} & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$\vec{a} = \begin{pmatrix} 1 \\ -\left(\frac{1}{2}+\frac{\sqrt{5}}{2}\right) \end{pmatrix}$ or better $\begin{pmatrix} 2 \\ -1-\sqrt{5} \end{pmatrix}$

For $\lambda = -\frac{1}{2} + \frac{\sqrt{5}}{2} > 0$ Get $\begin{pmatrix} 2 \\ -1+\sqrt{5} \end{pmatrix}$



(pplane demo)

4.6 Non-homogeneous syst

$\vec{x}\,' = A\vec{x}$ ← homog

$\vec{x}\,' = A\vec{x} + \vec{g}$ ← non-homog

EX: $\vec{x}\,' = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}\vec{x} + \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{2t}}_{\vec{g}}$

Step 1: Get homog sol^n $\vec{x}_c = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-2t} + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-4t}$

Gen^l Sol^n to non-homog: $\vec{x}_c + \vec{x}_p$

gen^l sol^n to homog

one particular sol^n to non-homog

Step 2: Try $\vec{x}_p = \vec{b}e^{2t}$ ← safe! 2 is not an e-val!

want

Plug in: $\vec{x}_p\,' = A\vec{x}_p + \vec{g}$

$$2\vec{b}e^{2t} = A\vec{b}e^{2t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{2t}$$ Cancel e^{2t}'s

$$2\vec{b} = A\vec{b} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$A\vec{b} - \underbrace{2\vec{b}}_{2I\vec{b}} = -\begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\underbrace{(\mathbb{A} - 2\mathbb{I})}_{\text{det} \neq 0 \text{ because } \underline{\text{not}} \text{ e-val}}\vec{b} = \begin{pmatrix}-1\\-2\end{pmatrix}$$

$$\left[\begin{array}{cc|c} \underbrace{-3-2}_{-5} & 1 & -1 \\ 1 & \underbrace{-3-2}_{-5} & -2 \end{array}\right]$$

Cramer's Rule:

$$b_1 = \frac{\det\begin{bmatrix}-1 & 1\\-2 & -5\end{bmatrix}}{\det\begin{bmatrix}-5 & 1\\1 & -5\end{bmatrix}} = \frac{7}{24}$$

$$b_2 = \frac{\det\begin{bmatrix}-5 & -1\\1 & -2\end{bmatrix}}{\det\begin{bmatrix}-5 & 1\\1 & -5\end{bmatrix}} = \frac{11}{24}$$

So $\vec{x}_p = \begin{pmatrix}7/24\\11/24\end{pmatrix}e^{2t}$

Gen' Sol^n $\quad \vec{x} = c_1\begin{pmatrix}1\\1\end{pmatrix}e^{-2t} + c_2\begin{pmatrix}1\\-1\end{pmatrix}e^{-4t} + \begin{pmatrix}7/24\\11/24\end{pmatrix}e^{2t}$

$$\begin{cases} x_1 = c_1e^{-2t} + c_2e^{-4t} + \frac{7}{24}e^{2t} \\ x_2 = c_1e^{-2t} - c_2e^{-4t} + \frac{11}{24}e^{2t} \end{cases}$$