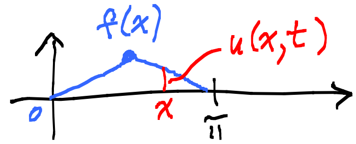
Lesson 25 on 11.1 Fourier Series $\quad$ 23, 24, 25 due Wed.

Harpsichord Prob:



$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} \quad (*)$$

(Take $c=1$)

I.C. $\begin{cases} u(x,0) = f(x) \\ \frac{\partial u}{\partial t}(x,0) = 0 \end{cases}$ $\qquad$ B.C. $\begin{cases} u(0,t) = 0 \\ u(\pi,t) = 0 \end{cases}$

Bernoulli v.s. Bernoulli $\qquad$ Jacob: $u(x,t) = \varphi(x+t) - \psi(x-t)$

Johann: Try $u(x,t) = X(x)T(t)$

Plug into $(*)$ $\qquad \frac{\partial^2 u}{\partial x^2} = X''(x)T(t) = X(x)T''(t) = \frac{\partial^2 u}{\partial t^2}$

Hmmm; Hold $x = \frac{\pi}{2}$. See $\frac{T''}{T}$ is const. Call it λ. Let x vary. See $\frac{X''}{X} \equiv \lambda$ too!

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} = \lambda$$

Get two ODEs

$$\begin{cases} X'' - \lambda X = 0 \\ T'' - \lambda T = 0 \end{cases}$$

Save I.C. for later. $\quad$ B.C.: $u(0,t) = X(0)T(t) = 0$ want

need $X(0) = 0$

$u(\pi,t) = X(\pi)T(t) = 0$

need $X(\pi) = 0$

$$T'' - \lambda T = 0$$

$$\begin{cases} X'' - \lambda X = 0 \\ X(0) = 0,\ X(\pi) = 0 \end{cases}$$

Sturm-Liouville Prob.

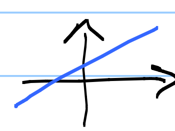
$$X'' = \lambda X$$

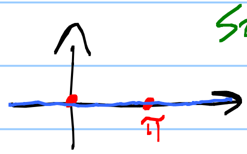
λ is an e-val for $L = \frac{d^2}{dx^2}$

Vector Sp. C^2-fcns satisfying BC's.

<u>Case $\lambda = 0$:</u> $X'' \equiv 0$

So $X(x) = c_1 x + c_2$, linear!





Ouch! $X(x) \equiv 0$

Case $\lambda > 0$: Write $\lambda = k^2$.

$$X'' - k^2 X = 0$$
$$r^2 - k^2 = 0$$
$$r = \pm k$$

$$X(x) = c_1 e^{kx} + c_2 e^{-kx}$$

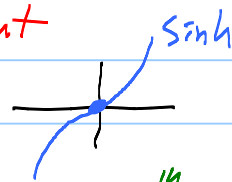
$$X(0) = c_1 e^0 + c_2 e^0 \overset{\text{want}}{=} 0$$

$$\boxed{c_2 = -c_1}$$

So $X(x) = c_1 e^{kx} + (-c_1) e^{-kx}$

$$= 2c_1 \sinh kx$$

$$X(\pi) = 2c_1 \underbrace{\sinh k\pi}_{\neq 0} \overset{\text{want}}{=} 0$$



Ouch! c_1 must be 0. Get zero sol^n!

Case $\lambda < 0$: $\lambda = -\mu^2$ $\qquad X'' + \mu^2 X = 0$

$$r^2 + \mu^2 = 0$$
$$r = \pm \mu i$$

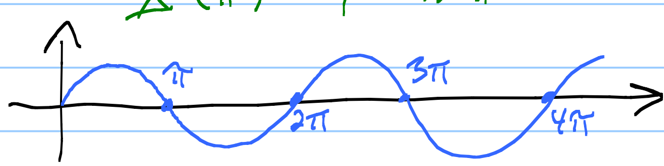
$$X(x) = c_1 \sin \mu x + c_2 \cos \mu x$$

$$X(0) = c_1 \cdot 0 + c_2 \cdot 1 \overset{\text{want}}{=} 0$$

$$\boxed{c_2 = 0}$$

So $X(x) = c_1 \sin \mu x$

$$X(\pi) = c_1 \sin \mu\pi \overset{\text{want}}{=} 0$$



Need $\mu\pi$ = multiple of π $= n\pi$, $n = 1, 2, 3, \ldots$

$\mu = n$. $\qquad \underline{\lambda = -\mu^2 = -n^2}$, $n = 1, 2, 3, \ldots$

Get non-zero sol^{n}s $\quad X_n(x) = \sin nx \quad \begin{pmatrix}\text{Take} \\ c_1 = 1\end{pmatrix}$

Get $T_n(t) = c_1 \cos nt + c_2 \sin nt$

Initial Cond: $\quad \frac{\partial u}{\partial t}(x, 0) \equiv 0$

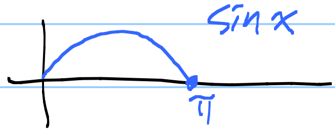
$$X_n(x)\,\underbrace{T_n'(0)}_{-nc_1 \sin n\cdot 0 + n c_2 \cos n\cdot 0 \,=\, 0} \equiv 0$$

want

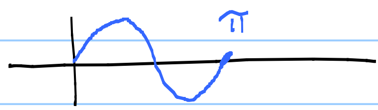
need $c_2 = 0$

Get $T_n(t) = \cos nt$ $\begin{pmatrix}\text{Take}\\ c_1 = 1\end{pmatrix}$

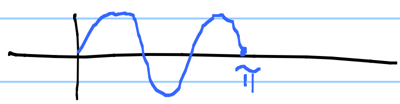
Get $\boxed{u_n(x,t) = \sin nx \cos nt}$

$n=1$ 

Cos t

$n=2$ 

Cos 2t

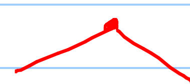
$n=3$ 

Cos 3t

PDE is linear! B.C. are homogeneous

Can add sol'ns to get more sol'ns!

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin nx \cos nt$$

Initial Cond? $u(x,0) = \sum_{n=1}^{\infty} c_n \sin nx \overset{?}{=}$ 

Yes! $\sin nx$ are orthogonal on $[0,\pi]$!

$$(\sin nx, \sin mx) = \int_0^{\pi} \sin nx \sin mx \, dx = \begin{cases} 0 & n \neq m \\ \frac{\pi}{2} & n = m \end{cases}$$

If $f(x) = \sum_{1}^{\infty} c_n \sin nx$

Mult. by $\sin mx$: $f(x)\sin mx = \sum_{1}^{\infty} c_n \sin nx \sin mx$

Integrate $\int_0^{\pi}$: $\int_0^{\pi} f(x) \sin mx \, dx = \sum_{1}^{\infty} c_n \underbrace{\int_0^{\pi} \sin nx \sin mx \, dx}_{=0}$

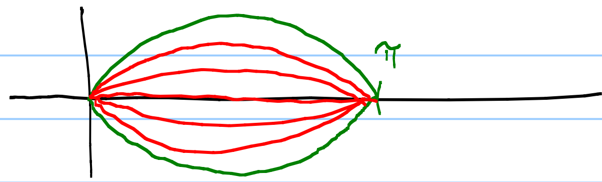
except when $n=m$

$$\int_0^{\pi} f(x) \sin mx \, dx = c_m \frac{\pi}{2}$$

$$\boxed{C_m = \frac{2}{\pi}\int_0^{\pi} f(x)\sin mx\, dx}$$

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin nx \cos nt$$

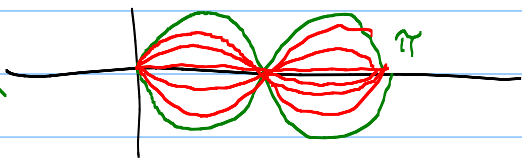
$n=1$



$\sin x \cos t$

pitch of string

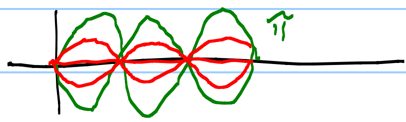
$n=2$



$\sin 2x \cos 2t$

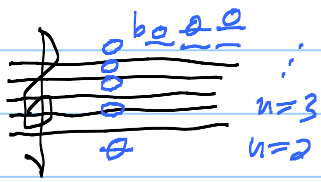
1 octave higher

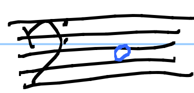
$n=3$



$\sin 3x \cos 3t$

octave and a fifth up





$n=1$

Euphonium, Tuba, Sax

All c's $\neq 0$

Trumpet, Clar,
Trombone

All odd c's $\neq 0$

even c's $= 0$