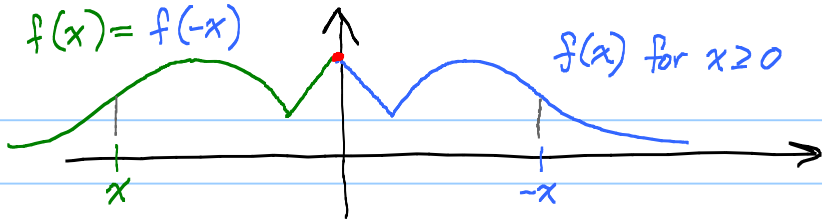
Lesson 31 on 11.8 Fourier Cosine & Sine Transforms

HWK 10: 29, 30, 31 due Wed



$$f(x) = \int_0^\infty A(w) \cos wx \, dw + \begin{pmatrix} B \text{ part} = 0 \\ \text{if } f \text{ even} \end{pmatrix}$$

where $A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos vw \, dv = \frac{2}{\pi} \int_0^\infty f(v) \cos vw \, dv$

Whoa! Do $f \longmapsto \int_0^\infty f(v) \cos vw \, dv$ twice, get $\frac{\pi}{2} f$ back!

Fourier Cosine Transform:

$$\tilde{\mathcal{F}}_c[f] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(v) \cos wv \, dv$$

fcn of w for $w > 0$

f fcn of x for $x > 0$

dummy variable

Amazing fact: $\tilde{\mathcal{F}}_c[\tilde{\mathcal{F}}_c[f]] = f$

Think: $\tilde{\mathcal{F}}_c$ is its own inverse!

Fourier Sine Transform: $\tilde{\mathcal{F}}_s[f] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin wx \, dx$

Notations: $\tilde{\mathcal{F}}_c[f] = \hat{f}_c \qquad \tilde{\mathcal{F}}_s[f] = \hat{f}_s$

Same amazing fact: $\tilde{\mathcal{F}}_s[\tilde{\mathcal{F}}_s[f]] = f$

Properties:
1) $\tilde{\mathcal{F}}_c[f'] = w \tilde{\mathcal{F}}_s[f] - \sqrt{\frac{2}{\pi}} f(0)$

2) $\tilde{\mathcal{F}}_s[f'] = -w \tilde{\mathcal{F}}_c[f]$

Note: In 1, 2 need f continuous and absolutely integrable, f' piecewise cont, and $f(x) \to 0$ as $x \to \infty$.

Why 1: $\tilde{\mathcal{F}}_c[f'] = \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \cos wx \, dx$

$$= \sqrt{\frac{2}{\pi}} \lim_{B\to\infty} \int_0^B \underbrace{\cos wx}_{u} \; \underbrace{f'(x)\,dx}_{dv}$$

$du = -w \sin wx \, dx \qquad v = f(x)$

$$= \sqrt{\frac{2}{\pi}} \lim_{B\to\infty} \left[(\cos wx) f(x) \Big|_0^B - \int_0^B f(x) \underbrace{(-w \sin wx \, dx)}_{du} \right]$$

$$= \sqrt{\frac{2}{\pi}} \lim_{B\to\infty} \left(\left[(\underbrace{\cos wB}_{\text{bounded}}) \underbrace{f(B)}_{\to 0 \text{ as } B\to\infty} - 1 \cdot f(0) \right] + w \underbrace{\int_0^B f(x) \sin wx \, dx}_{\xrightarrow[B\to\infty]{} \tilde{\mathcal{F}}_s[f]\sqrt{\frac{\pi}{2}}} \right)$$

$$= -\sqrt{\frac{2}{\pi}} f(0) + w \tilde{\mathcal{F}}_s[f] \checkmark$$

Consequence: a) $\tilde{\mathcal{F}}_c[f''] = -w^2 \tilde{\mathcal{F}}_c[f] - \sqrt{\frac{2}{\pi}} f'(0)$

b) $\tilde{\mathcal{F}}_s[f''] = -w^2 \tilde{\mathcal{F}}_s[f] + \sqrt{\frac{2}{\pi}} w f(0)$

Prob: Find $\tilde{\mathcal{F}}_c[e^{-ax}]$

$f(x) = e^{-ax}$

$f'(x) = -ae^{-ax} \leftarrow f'(0) = -a$

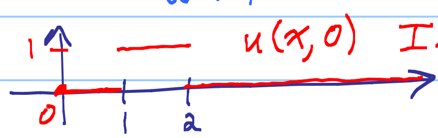
$f''(x) = a^2 e^{-ax}$

Aha! Use (a): $\tilde{\mathcal{F}}_c[\underbrace{f''}_{a^2e^{-ax}}] = -w^2 \tilde{\mathcal{F}}_c[\underset{e^{-ax}}{f}] - \sqrt{\frac{2}{\pi}} \underbrace{f'(0)}_{-a}$

$$a^2 \tilde{\mathcal{F}}_c[e^{-ax}] = -w^2 \tilde{\mathcal{F}}_c[e^{-ax}] + a\sqrt{\frac{2}{\pi}}$$

Solve for $\tilde{\mathcal{F}}_c$: $\tilde{\mathcal{F}}_c[e^{-ax}] = \dfrac{a\sqrt{\frac{2}{\pi}}}{w^2+a^2}$

Application: Hot wire



I.C.

Heat Eqn: $u_{xx} = u_t$ [B.C. Ice at zero

$\llcorner\ u(0,t)=0$ for all t

Apply $\tilde{\mathcal{F}}_s$ in space variable.

Write $\hat{u}(w,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty u(x,t) \sin wx \, dx$

Important step: Can put $\frac{\partial}{\partial t}$ under $\int$ in

$$\frac{\partial \hat{u}}{\partial t}(w,t) = \tilde{\mathcal{F}}_s\left[\frac{\partial u}{\partial t}\right]$$

But:

$$\tilde{\mathcal{F}}_s\left[\frac{\partial^2 u}{\partial x^2}\right] = -w^2 \underbrace{\tilde{\mathcal{F}}_s[u]}_{\hat{u}} + \sqrt{\frac{2}{\pi}}\, w\, \underbrace{u(0,t)}_{=0 \text{ by B.C.}}$$

Hit PDE with $\tilde{\mathcal{F}}_s$ in x var:

$$-w^2 \hat{u}(w,t) = \frac{\partial \hat{u}}{\partial t}(w,t)$$ exp decay!

Sol^n:

$$\hat{u}(x,t) = C(w) e^{-w^2 t}$$

constant diff for diff w's

Let $t=0$ to find $C(w)$:

$$\hat{u}(x,0) = C(w)\, e^{-w^2 \cdot 0} = C(w)$$

So $C(w) = \tilde{\mathcal{F}}_s[\text{Initial Temp}]$

$$= \sqrt{\frac{2}{\pi}} \int_1^2 1 \cdot \sin wx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[-\frac{1}{w}\cos wx\right]_{x=1}^{2}$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{-\cos 2w - (-\cos w)}{w}\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\cos w - \cos 2w}{w}\right]$$

nice at $w=0$. L'H

So $\hat{u}(w,t) = \sqrt{\frac{2}{\pi}}\left[\frac{\cos w - \cos 2w}{w}\right]e^{-w^2 t}$

Finally, undo $\tilde{\mathcal{F}}_s$ by apply $\tilde{\mathcal{F}}_s$ again!

$$u(x,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\left(\frac{\cos w - \cos 2w}{w}\right)e^{-w^2 t}\sin wx \, dw$$

The complex version of Fourier Series:

$$\cos x = \frac{e^{ix}+e^{-ix}}{2} \qquad \sin x = \frac{e^{ix}-e^{-ix}}{2i}$$

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos nx + b_n \sin nx\right)$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$

$\frac{e^{inx}}{\sqrt{2\pi}}$ orthonormal basis for $L^2[-\pi,\pi]$

complex

Lesson 32 on 11.9. On Exam 2, but Lesson 32 HWK not to be turned in.

Read only 11.9: pages 522-528.

Skip DFT, FFT.