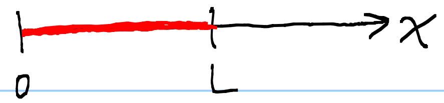
Lesson 38



Hot wire with insulated ends

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Heat Eqn

BC

$$\frac{\partial u}{\partial x}(0,t) = 0$$

$$\frac{\partial u}{\partial x}(L,t) = 0$$

IC

$$u(x,0) = f(x) \leftarrow \text{given}$$

Separation of variables: Try $u(x,t) = X(x)T(t)$

$$XT' = c^2 X'' T \qquad \frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T} = \lambda$$

X-prob: $X'' - \lambda X = 0$
$$\begin{cases} X'(0) = 0 \\ X'(L) = 0 \end{cases}$$

T-prob: $T' - \lambda c^2 T = 0$

Case $\lambda = 0$: $X'' \equiv 0$, $X(x) = c_1 x + c_2$

$X'(0) = c_1 \overset{\text{want}}{=} 0$ Take $c_2 = 1$.

$X'(L) = c_1 = 0$ Get $X_0(x) \equiv 1$.

$\lambda = 0$ is an e-val with e-fcn $X_0 = 1$.

Case $\lambda > 0$: $\lambda = \mu^2$ $X'' - \mu^2 X = 0$ $r^2 - \mu^2 = 0$

$r = \pm\mu$

$X(x) = c_1 \cosh\mu x + c_2 \sinh\mu x$ or $k_1 e^{\mu x} + k_2 e^{-\mu x}$

$X'(x) = c_1\mu \sinh\mu x + c_2\mu\cosh\mu x$

$$\text{BC}\begin{cases} X'(0) = c_1 \cdot 0 + c_2\mu \cdot 1 \overset{\text{want}}{=} 0 \\ X'(L) = \underset{\neq 0}{c_1\mu}\,\underbrace{\sinh\mu L}_{\neq 0} = 0 \end{cases}$$

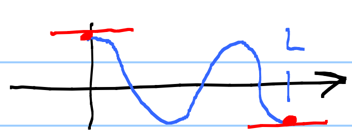
$\boxed{c_2 = 0}$

$c_1 = 0$ too!

No non-zero sol^n if $\lambda > 0$.

Case $\lambda < 0$:

$\lambda = -\mu^2$



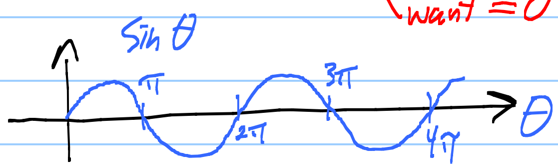
$X'' + \mu^2 X = 0 \qquad r^2 + \mu^2 = 0 \qquad r = \pm \mu i$

$X(x) = c_1 \cos \mu x + c_2 \sin \mu x$

$X'(x) = -c_1 \mu \sin \mu x + c_2 \mu \cos \mu x$

BC: $X'(0) = -c_1 \mu \cdot 0 + c_2 \mu \cdot 1 \overset{\text{want}}{=} 0 \quad \boxed{c_2 = 0}$

$X'(L) = -c_1 \mu \underbrace{\sin \mu L}_{\text{want } = 0 \text{ so } c_1 \neq 0 \text{ is allowed.}} \overset{\text{want}}{=} 0$



Need $\mu L = n\pi$

$n = 1, 2, 3, \dots$

Get e-vals $\lambda = -\mu^2 = -\left(\frac{n\pi}{L}\right)^2 \quad n = 1, 2, 3, \dots$

and e-fcns $X_n(x) = \cos \frac{n\pi}{L} x \quad \boxed{\text{Take } c_1 = 1}$

Superposition: $u(x,t) = c_0 X_0(x) T_0(t) + \sum_{n=1}^{\infty} c_n X_n(x) T_n(t)$

$\underline{\lambda = 0}$: $T' \equiv 0$. $T(t) = C$. Take $C = 1$.

$T_0(t) = 1$.

$\underline{\lambda_n = -\left(\frac{n\pi}{L}\right)^2}$: $T' = -\left(\frac{n\pi}{L}\right)^2 c^2 T \qquad T = K e^{-\left(\frac{n\pi c}{L}\right)^2 t}$

Take $K = 1$.

$T_n(t) = e^{-\left(\frac{n\pi c}{L}\right)^2 t}$

$$u(x,t) = c_0 + \sum_{n=1}^{\infty} c_n \cos \frac{n\pi x}{L} \, e^{-\left(\frac{n\pi c}{L}\right)^2 t}$$

Finally, need IC: $u(x,0) = c_0 + \sum_{n=1}^{\infty} c_n \cos \frac{n\pi x}{L} \overset{\text{want}}{=} f(x)$

Fourier Cosine Series!

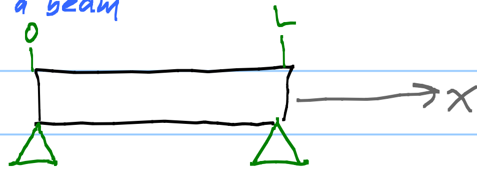
Need

$c_0 = \frac{1}{L} \int_0^L f(x)\, dx \leftarrow$ Average initial temp

$c_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L}\, dx \qquad n = 1, 2, \dots$

Interesting fact: See $\lim_{t\to\infty} u(x,t) = C_0 \leftarrow$ Ave temp

p. 551: 15, 16. Vibrations of a beam



$$\frac{\partial^2 u}{\partial t^2} = -c^2 \frac{\partial^4 u}{\partial x^4}$$

BC

$u(0,t)=0$ $\quad$ $\frac{\partial^2 u}{\partial x^2}(0,t)=0$

$u(L,t)=0$ $\quad$ $\frac{\partial^2 u}{\partial x^2}(L,t)=0$

simply supported $\quad$ not clamped, so zero curvature at endpoints

IC

$u(x,0)=f(x)$

$\frac{\partial u}{\partial t}(x,0)=g(x)$

#16. Starting from "rest" means $g(x)\equiv 0$.

Try $u(x,t)=X(x)T(t)$:

$$\frac{X^{(4)}}{X} = -\frac{1}{c^2}\frac{T''}{T} = \lambda, \text{ a const.}$$

X-prob:

$$\boxed{\begin{gathered} X^{(4)} - \lambda X = 0 \\ X(0)=0 \quad X''(0)=0 \\ X(L)=0 \quad X''(L)=0 \end{gathered}}$$

Case $\lambda=0$: $X^{(4)}\equiv 0$. $\quad X(x)=c_1+c_2x+c_3x^2+c_4x^3$

$X''=2c_3+6c_4x$

$X(0)=c_1 \overset{\text{want}}{=} 0$

$X''(0)=2c_3=0$

$\boxed{c_1=0,\ c_3=0}$

$X(L)=c_2L+c_4L^3 \overset{\text{want}}{=} 0 \leftarrow$ ouch! $c_2=0$ too!

$X''(L)=6c_4L=0$

$\boxed{c_4=0}$

Only get zero sol^n!

Case $\lambda>0$: $\leftarrow$ only do this case for 15, 16

$\lambda = \beta^4 \qquad X^{(4)} - \beta^4 X = 0 \qquad r^4 - \beta^4 = 0$

$$(r^2-\beta^2)(r^2+\beta^2) = 0$$

$$(r-\beta)(r+\beta)(r^2+\beta^2) = 0$$

$$r = \pm\beta, \pm\beta i$$

$$X(x) = c_1 \cos\beta x + c_2 \sin\beta x + \underbrace{c_3 \cosh\beta x + c_4 \sinh\beta x}_{\text{or } k_3 e^{\beta x} + k_4 e^{-\beta x}}$$

Prob: Find values of β that allow non-zero sol'ns with BC. Get $X_n(x)$. Then get T_n

Finally, use $u(x,t) = \sum_{n=1}^{\infty} X_n(x)\left(A_n T_{n,1}(t) + B_n T_{n,2}(t)\right)$

I.C. Given $f(x)$. $g(x) \equiv 0$.

Case $\lambda < 0$: $\Leftarrow$ don't do this case.

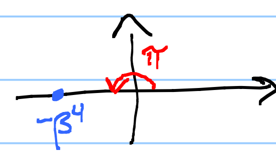
$\lambda = -\beta^4 \qquad X^{(4)} + \beta^4 X = 0 \qquad r^4 + \beta^4 = 0$

$$r^4 = -\beta^4$$

4 complex roots

Look for roots $z = Re^{i\theta}$:

Need $\quad (Re^{i\theta})^4 = -\beta^4$



$$R^4 e^{i4\theta} = \beta^4 e^{i\pi}$$

$\leftarrow$ angles must represent same spot in $\mathbb{C}$

radii must be =

Get $\quad R = \beta$

$4\theta = \pi + 2n\pi \qquad n = 0, \pm1, \pm2,$

$\theta = \frac{\pi}{4} + \frac{n\pi}{2}$