Lesson 2 MA 527

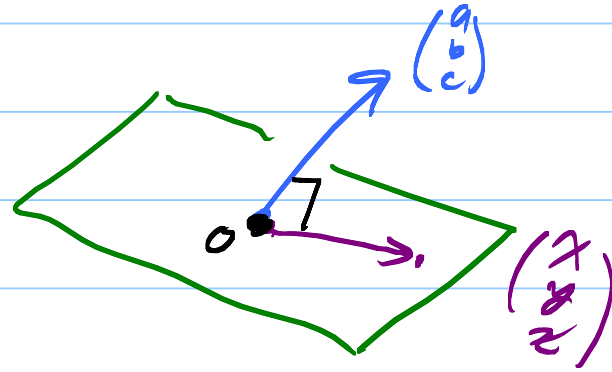
$A\vec{x} = \vec{0}$ ← homogeneous (always has $\vec{x} = \vec{0}$ sol^n)

$A\vec{x} = \vec{b}$ ← non-homog (might have no sol^{n}s, ∞ many sol^{n}s, a unique sol^n)

See p. 274 for 3-D pic

$ax + by + cz = 0$

$(a, b, c) \cdot (x, y, z) = 0$



Gaussian Elimination

$$\begin{cases} 2x_1 + x_2 - x_3 = 3 \\ x_1 - x_2 + 2x_3 = 1 \\ -4x_1 + 6x_2 - 7x_3 = 1 \\ 2x_1 \qquad + x_3 = 3 \end{cases}$$

overdetermined
too many eqns.

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ 1 & -1 & 2 & 1 \\ -4 & 6 & -7 & 1 \\ 2 & 0 & 1 & 3 \end{array}\right]$$

High School rules:

1) Can swap eqns.
2) Can multiply an eqn by $c \neq 0$.
3) Can multiply one eqn by $c \neq 0$ and add it to another.

College rules: Change word eqns to rows and operate on the matrix.

HS: Eliminate vars.

C: Reduce matrix to Row Echelon form.

Do back substitution.

EX:

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 8 \\ 0 & 5 & 6 & 9 \\ 0 & 0 & 7 & 10 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

dangerous spot!

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &= 8 \quad \text{E1} \\ 5x_2 + 6x_3 &= 9 \quad \text{E2} \\ 7x_3 &= 10 \quad \text{E3} \\ 0 &= 0 \end{aligned}$$

whew!

Signature of an inconsistent system:

$$\left[\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 3 \end{array}\right] \leftarrow \text{ouch!}$$

Back substitution: E3: $x_3 = 10/7$

E2: $5x_2 + 6\left(\frac{10}{7}\right) = 9$ ← Get x_2.

E1: $2x_1 + 3x_2 + 4x_3 = 8$ Get x_1.

$$\begin{array}{cc} x_2 & x_3 \\ \uparrow & \uparrow \\ \text{know} & 10/7 \end{array}$$

EX:

$$\begin{cases} x_1 - x_2 + x_3 - 2x_4 = -1 \\ 2x_1 - 2x_2 + x_3 - 2x_4 = -3 \\ -x_1 + x_2 - 2x_3 + 4x_4 = 0 \end{cases}$$

underdetermined
too many vars.

$$\begin{array}{cl} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & -2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] & \begin{array}{l} E1 \\ E2 \\ \leftarrow \text{whew!} \end{array} \\ \begin{array}{cccc} \uparrow & & \uparrow & \\ x_1 \text{ Col} & & x_3 \text{ Col} & \end{array} & \end{array}$$

x_1, x_3 are <u>bound</u> variables

x_2, x_4 are <u>free</u> variables. ← ∞ many sol^{n}s

Method: **Step 1:** Let free be parameters:

$$x_2 = t_1 \qquad x_4 = t_2$$

free to vary.

Step 2: Plug parameters in and solve for the bound vars.

$$x_1 - \underbrace{\boxed{t_1}}_{x_2} = -2 \qquad E1$$

$$x_3 - 2\underbrace{\boxed{t_2}}_{x_4} = 1 \qquad E2$$

$$\begin{cases} x_1 = -2 + t_1 \\ x_3 = 1 + 2t_2 \end{cases} \qquad (x_1, x_3 \text{ bound})$$

Step 3: Make a list of vars.

$$\begin{cases} x_1 = -2 + t_1 \\ x_2 = t_1 \\ x_3 = 1 + 2t_2 \\ x_4 = t_2 \end{cases}$$

Step 4: Pull it apart:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \underset{\uparrow \text{ point in } \mathbb{R}^4}{\begin{pmatrix} -2 \\ 0 \\ 1 \\ 0 \end{pmatrix}} + \underbrace{t_1 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 0 \\ 0 \\ 2 \\ 1 \end{pmatrix}}_{\text{2-D plane}}$$

$$\underset{\substack{\uparrow \\ \text{particular} \\ \text{sol}^n \text{ to } A\vec{x} = \vec{b}}}{\vec{x}_p} + \underbrace{\left(t_1 \vec{x}_1 + t_2 \vec{x}_2 \right)}_{\substack{\text{Gen}^l \text{ Sol}^n \text{ to} \\ A\vec{x} = \vec{0}}}$$

The facts: $A\vec{x} = \vec{b}$

$A^{\#} = [A \mid \vec{b}]$ ← augmented matrix

$\rightsquigarrow [E \mid \vec{e}]$ ← Row echelon form.

$$\text{Rank}(A) = \#\text{ non-zero rows in } E$$
$$= \#\text{ bound vars}$$

$$\text{Rank}(A^{\#}) = \#\text{ non-zero rows in } [E \mid \vec{e}].$$

1) $\mathrm{Rank}(A) < \mathrm{Rank}(A^\#)$

no solutions.

2) $\mathrm{Rank}(A) = \mathrm{Rank}(A^\#)$ ← consistent

a) $\mathrm{Rank}(A) = \#\text{vars}$.

all vars are <u>bound</u>.

Do back sub and get

unique sol^n.

b) $\mathrm{Rank}(A) < \#\text{vars}$.

free vars exist

∞ many sol^{n}s.

<u>Homogeneous case</u>: $\vec{b} = \vec{0}$

$$[A \mid \vec{0}] \rightsquigarrow [E \mid \vec{0}]$$

Always consistent.

a) $\mathrm{Rank}(A) = \#\text{vars}$

Unique sol^n: $\vec{x} = \vec{0}$

b) $\text{Rank}(A) < \#\text{ vars}$

∞ many sol^{n}s.

$$\begin{bmatrix} 3 & 2 & 1 & 0 \\ 1 & 2 & 3 & 0 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

$$\begin{array}{r} \\ \text{kill} \\ \text{kill} \end{array} \begin{bmatrix} 1 & 2 & 3 & 0 \\ 3 & 2 & 1 & 0 \\ 5 & 6 & 7 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 23 & 11 & 13 \\ 0 & 5 & 6 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 0 & 5 & 6 & 7 \\ 0 & 23 & 11 & 13 \end{bmatrix} \quad -\tfrac{23}{5}E2 + E3$$

kill

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 0 & 5 & 6 & 7 \\ 0 & 0 & 87 & 3 \end{array}\right]$$

Row echelon

<u>not unique</u>

$$\left[\begin{array}{ccc|c} 1 & 2 \leftarrow \text{kill} & 3 \leftarrow \text{kill} & \\ 0 & 1 & 6/5 \leftarrow \text{kill} & 7/5 \\ 0 & 0 & 1 & 3/87 \end{array}\right]$$

Use 1's to clean out above.

Get Reduced Row Echelon form.

It is unique.