Lesson 3 MA 527 7.4

HWK 1 Lessons 1,2,3 from syllabus due Wed. 8/28

On-campus: Paper, stapled, beginning of class.

Off-campus: Scanned PDF upload on Blackboard

https://mycourses.purdue.edu

BellSteveHWK1.pdf

EX: $[A \mid \vec{b}] \sim$ reduced r.e. form

$$\begin{array}{c} \\ \\ \\ \\ \\ \end{array}\left[\begin{array}{cccc|c} 5 & -2 & 0 & -3 & 6 \\ 0 & 0 & 3 & 4 & 7 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]\begin{array}{l} E1 \\ E2 \\ \\ \end{array}$$

x_1 bound, x_2 free, x_3 bound, x_4 free

$\text{Rank}(A) = 2$

free

$$\boxed{\begin{aligned} x_2 &= t_1 \\ x_4 &= t_2 \end{aligned}}$$

E1: $x_1 = \frac{6}{5} + \frac{2}{5}\underbrace{\boxed{t_1}}_{x_2} + \frac{3}{5}\underbrace{\boxed{t_2}}_{x_4}$

E2: $x_3 = \frac{7}{3} - \frac{4}{3}\underbrace{\boxed{t_2}}_{x_4}$

$$\begin{cases} x_1 = \frac{6}{5} + \frac{2}{5}t_1 + \frac{3}{5}t_2 \\ x_2 = t_1 \\ x_3 = \frac{7}{3} - \frac{4}{3}t_2 \\ x_4 = t_2 \end{cases}$$

$$\vec{x} = \begin{pmatrix} 6/5 \\ 0 \\ 7/3 \\ 0 \end{pmatrix} + t_1\begin{pmatrix} 2/5 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t_2\begin{pmatrix} 3/5 \\ 0 \\ -4/3 \\ 1 \end{pmatrix}$$

$$\vec{x} = \underset{\substack{\uparrow \\ \text{part.} \\ \text{sol}^n}}{\vec{x}_p} + \underbrace{\left(t_1 \vec{v}_1 + t_2 \vec{v}_2 \right)}_{\substack{\text{Gen}^l \text{ Solution to} \\ A\vec{x} = 0}}$$

Null space of $A = \{ \vec{x} : A\vec{x} = 0 \}$.

Nullity of A = Dimension of null space.
= # free vars.

Big fact: $\underbrace{(\#\text{ bound vars})}_{\text{Rank}(A)} + \underbrace{(\#\text{ free vars})}_{\text{nullity}(A)} = \#\text{vars.}$

$\text{Rank}(A) + \text{nullity}(A) = \#\text{ cols } A$

Def^n: $V = \text{span}(\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n)$

$$= \{ \underbrace{c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n}_{\text{linear combo}} : c\text{'s} \in \mathbb{R} \}$$

$\mathbb{R}^n$ is a vector space. V is too.

Def^n: W is a subspace of V means

1) $W \subset V$

2) W is a vector space.

<u>Subspace Test</u>: 1) If $\vec{w}_1$ and $\vec{w}_2$ in W, then so is $\vec{w}_1 + \vec{w}_2$.

2) If $\vec{w} \in W$, then so is $c\vec{w}$.

<u>EX</u>: $W = \left\{ \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} : a, b \in \mathbb{R} \right\}$.

1, 2 Easy. W is a subspace of $\mathbb{R}^3$.

<u>EX</u>: $W = \left\{ \begin{pmatrix} a \\ b \\ 13 \end{pmatrix} : a, b \in \mathbb{R} \right\}$.

is <u>not</u>.

<u>EX</u>: $\text{Null}(A) = \{\vec{x} : A\vec{x} = \vec{0}\}$ is a subspace of $\mathbb{R}^n$.

$\mathbb{R}^3$: $$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = x \underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{\vec{e}_1} + y \underbrace{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}}_{\vec{e}_2} + z \underbrace{\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}_{\vec{e}_3}$$

$\vec{e}_1, \vec{e}_2, \vec{e}_3$ span $\mathbb{R}^3$. They are <u>independent</u>. So they form a <u>basis</u> for $\mathbb{R}^3$.

$\mathbb{R}^3$ is 3-Dimensional.

<u>Def^n</u>: $\vec{v}_1, \ldots, \vec{v}_n$ are <u>linearly independent</u> if whenever $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n = \vec{0}$, <u>all</u> the c's <u>must</u> be zero.

<u>Question</u>: Are $\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix}$ lin indep?

Suppose $c_1\begin{pmatrix}1\\2\\3\end{pmatrix} + c_2\begin{pmatrix}4\\5\\6\end{pmatrix} + c_3\begin{pmatrix}7\\8\\9\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

$$\begin{bmatrix}1 & 4 & 7\\2 & 5 & 8\\3 & 6 & 9\end{bmatrix}\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = \vec{0}$$

Solve $\rightsquigarrow \left[\begin{array}{ccc|c}1 & 0 & -1 & 0\\0 & 1 & 2 & 0\\0 & 0 & 0 & 0\end{array}\right]$

x_3 free

∞ many sol^{n}s.
possible for non-zero c's.

Not indep! They are <u>dependent</u>.

<u>Better way</u>: How to decide if

$\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$ are indep <u>and</u> find

a basis for span.

<u>Step 1</u>: Stick them in as the <u>rows</u>

of a matrix $\mathbb{A}$.

<u>Step 2</u>: Row reduce to row echelon form $\mathbb{E}$.

<u>Step 3</u>: If the bottom is <u>not</u> all zeroes, they are <u>independent</u>. If the bottom row is all zeroes, they are dependent.

Bonus: The non-zero rows in $\mathbb{E}$ form a basis for the span.

<u>EX</u>: Are $\begin{pmatrix}1\\4\\7\end{pmatrix}, \begin{pmatrix}2\\5\\8\end{pmatrix}, \begin{pmatrix}3\\6\\9\end{pmatrix}$ indep?

1. $\mathbb{A} = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$

2. $A \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \leftarrow$ darn!

Not independent.

Bonus: span of my 3 vectors =

$$\text{span}\left(\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \right)$$

turn non-zero rows in E back to col vects.

This method yields linearly independent spanning vectors. They form a basis for the span. Dim (span) = Rank (A).

Why it works: Big fact: row operations do not change the span.

Great fact: $\text{Rank}(A) = \text{Rank}(A^T)$

Consequently: Can stick $\vec{v}_1, \ldots, \vec{v}_n$ in as columns of a matrix B.

If $\text{Rank}(\mathbb{B}) = n$, then they are <u>independent</u>. $\text{Rank}(\mathbb{B}) < n$, then they are dependent.