Lesson 5 7.7, 7.8

Homework 2: Lessons 4,5,6 due Wed after Labor Day

$$\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = (+1)\, a \det\begin{bmatrix} e & f \\ h & i \end{bmatrix} + (-1)\, b \det\begin{bmatrix} d & f \\ g & i \end{bmatrix} + (+1)\, c \det\begin{bmatrix} d & e \\ g & h \end{bmatrix} \leftarrow \text{row } 1$$

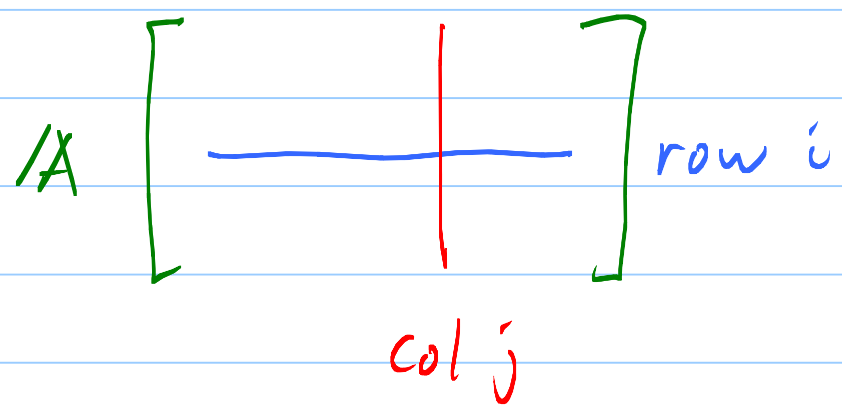
$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$= (-1)\, b \det\begin{bmatrix} d & f \\ g & i \end{bmatrix} + (+1)\, e \det\begin{bmatrix} a & c \\ g & i \end{bmatrix} + (-1)\, h \det\begin{bmatrix} a & c \\ d & f \end{bmatrix} \leftarrow \text{col } 2$$

MA 527: $\det(A_{n\times n})$ defined inductively:

$\det([a_{11}]) = a_{11}$, $\det(n\times n)$ = expand along row 1 to reduce to previous size.

Words:



A_{ij} = matrix from A by crossing out row i and col j.

$M_{ij} = \det(A_{ij})$ = "ij minor"

$$\begin{bmatrix} + & - & + & \cdots \\ - & + & - & \cdots \\ \vdots & \vdots & & \\ & & & \end{bmatrix} = [e_{ij}] \qquad e_{ij} = (-1)^{i+j}$$

$C_{ij} = (-1)^{i+j} M_{ij}$ = "ij cofactor"

__Fact:__ $\det(\underset{n \times n}{A}) = \sum_{j=1}^{n} a_{ij} C_{ij}$ ← i fixed, "expand along row i"

$= \sum_{i=1}^{n} a_{ij} C_{ij}$ ← j fixed "expand along col j"

__Properties of det:__ 1) $\det(I) = \underline{1}$

2) Switching rows changes sign of det

3) Adding c times one row to another row does __NOT__ change det

4) Multiplying a row by c multiplies det by c. [Can factor out

constants from rows.]

5) $\det(A^T) = \det(A)$

6) #5 $\Rightarrow$ 2,3,4 have column versions.

7) $\det(\underset{n \times n}{A}\underset{n \times n}{B}) = \det(A)\det(B)$

if and only if

8) $\det(\underset{n \times n}{A}) = 0$ iff $\underbrace{\text{Rank}(A) < n}$

$A\vec{x} = \vec{0}$ has ∞ many sol^{n}s

rows of A dependent

9) $\det(\underset{n \times n}{A}) \neq 0$ iff $\underbrace{\text{Rank}(A) = n}$

$A\vec{x} = \vec{0}$ has only $\vec{x} = 0$ solution

rows of A indep

Adult way to calculate $\det(A)$:

$$\det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 3 & 1 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & -5 & -8 \end{bmatrix} \begin{matrix} \\ \\ -3R_1 + R_3 \end{matrix}$$

kill

kill

$$= \det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & -7/4 \end{bmatrix} \quad \tfrac{5}{4}R_2 + R_3$$

← upper triangular

$$= 1 \cdot 4 \cdot \left(-7/4\right) = -7$$

Fact: $\det(\Delta \text{ matrix}) =$ product down diagonal

Why: $\det \begin{bmatrix} a & * & * \\ 0 & b & * \\ 0 & 0 & c \end{bmatrix} = a \det \begin{bmatrix} b & * \\ 0 & c \end{bmatrix} = ab \det[c]$

$= abc$

col 1   col 1

Inverse of a matrix: $\underset{n \times n}{A} \left[\vec{x}_1, \dots, \vec{x}_n\right] = \underbrace{\begin{bmatrix} 1 & 0 & & 0 \\ 0 & 1 & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}}_{I}$

$A\vec{x}_j = \vec{e}_j$ ← jth col of I

$$[A \mid e_j] \rightsquigarrow [I \mid \vec{y}_j]$$

rank $A = n$

$$\begin{cases} x_1 = y_1 \\ x_2 = y_2 \\ \vdots \\ x_n = y_n \end{cases}$$

Aha! sol^n $\vec{x} = \vec{y}_j$

Aha! $[A \mid I] \sim [I \mid B]$

$$\underset{B = A^{-1}.}{\overset{\uparrow}{B}}$$

EX:

$$\begin{array}{c} \begin{array}{ccc|ccc} & A & & & I & \end{array} \\ \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 10 & 0 & 0 & 1 \end{array}\right] \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2/3 & -4/3 & 1 \\ 0 & 1 & 0 & -2/3 & 11/3 & -2 \\ 0 & 0 & 1 & 1 & -2 & 1 \end{array}\right]$$

$$\underbrace{\quad\quad}_{I} \qquad \underbrace{\quad\quad\quad}_{A^{-1}}$$

<u>So far:</u> $A \cdot A^{-1} = I$

Hmmm. $[A \mid I] \sim [I \mid A^{-1}]$

$[I \mid A] \backsim$

Switch matrices and do the reverse of operations I just went through.

Wow! $(A^{-1})^{-1} = A$. This means that

$$A^{-1} \cdot A = I \quad \text{too!}$$

$$\boxed{A \cdot A^{-1} = A^{-1} \cdot A = I} \quad \text{unusual!}$$

Using $\mathbb{A}^{-1}$ to solve $\mathbb{A}\vec{x} = \vec{b}$:

$$\mathbb{A}^{-1}(\mathbb{A}\vec{x}) = \mathbb{A}^{-1}\vec{b} \quad \leftarrow \text{left multiply}$$

$$\underbrace{(\mathbb{A}^{-1}\mathbb{A})}_{\mathbb{I}}\vec{x} =$$

$$\boxed{\vec{x} = \mathbb{A}^{-1}\vec{b}} \quad \leftarrow \text{expensive!}$$

Famous Formula for $\mathbb{A}^{-1}$:

$$\mathbb{A}^{-1} = \frac{1}{\det(\mathbb{A})} \underbrace{\begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & & & \vdots \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{bmatrix}^T}_{\text{classical adjoint}}$$

$c_{ij} = ij$ cofactor

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

"switch diag elements, make rest minuses, mult by $\frac{1}{\det}$"

Cramer's Rule: $\underset{n\times n}{\mathbb{A}}\vec{x} = \vec{b}$, $\text{Rank}(\mathbb{A}) = n$.

Let $\mathbb{A}_j$ = matrix gotten from $\mathbb{A}$ by replacing col j by $\vec{b}$.

Sol^n $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$: $x_j = \dfrac{\det(\mathbb{A}_j)}{\det(\mathbb{A}) \leftarrow \neq 0}$

Why: $\mathbb{A} = [\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n]$

$$\mathbb{A}\vec{x} = [\vec{a}_1, \ldots, \vec{a}_n]\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \vec{b}$$

$$\boxed{x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_n\vec{a}_n = \vec{b}}$$

$$x_1 \det(\mathbb{A}) = \det[x_1\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n]$$

$x_2\vec{a}_2 + \text{col } 1$

$$= \det[x_1\vec{a}_1 + x_2\vec{a}_2, \vec{a}_2, \ldots, \vec{a}_n]$$

= continue =

$$= \det\Big[\underbrace{x_1\vec{a}_1 + \cdots x_n\vec{a}_n}_{\vec{b}}, \vec{a}_2, \ldots, \vec{a}_n\Big]$$

$$= \det(A_1) \qquad x_1 = \frac{\det(A_1)}{\det(A)} \checkmark$$

Repeat for $x_j \det(A) = \cdots$.

Words: $\det(A) = 0$ Bad! A singular

$\mathrm{Rank}(A) < n$, rows dependent.

$\det(A) \neq 0$ Good! A non-singular

$\mathrm{Rank}(A) = n$, rows indep.