Lesson 6 7.9 Vector Spaces Lessons 4,5,6 due Wed

EX: $y'' - y = 0$ $\qquad L[y] = y'' - y$

linear transformation

Gen^l Sol^n $\quad y = c_1 e^x + c_2 e^{-x} \leftarrow \text{span}\{e^x, e^{-x}\}$

$C^2(\mathbb{R})$ = twice continuously diff'ble fcns on $\mathbb{R}$

vector space. Rules: Box on page 310.

[add vectors, mult. by c, -vectors, 0 vector]

Are e^x and e^{-x} lin. indep.?

Suppose $c_1 e^x + c_2 e^{-x} = 0$

zero function

Case $c_1 \neq 0$: $\quad c_1 e^x = -c_2 e^{-x}$

$$e^x \cdot e^x = -\frac{c_2}{c_1} e^{-x} \cdot e^x$$

$$e^{2x} = \frac{-c_2}{c_1} \leftarrow \text{no way!}$$

Case $c_1 = 0$: $\quad 0 \cdot e^x + c_2 e^{-x} = 0$

$$c_2 e^{-x} = 0$$

e^a is never zero.

Conclude $c_2 = 0$ too.

Both c_1 and c_2 must be zero.

e^x, e^{-x} independent. $\{e^x, e^{-x}\}$ form a <u>basis</u> for sol^{n}s.

<u>Fancier way:</u>

$$\begin{cases} c_1 e^x + c_2 e^{-x} = 0 & (A) \\ \frac{d}{dx}(A): \quad c_1 e^x - c_2 e^{-x} = 0 & (B) \end{cases}$$

$$\begin{bmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{bmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \vec{0}$$

$$\det = e^x \cdot (-e^{-x}) - e^{-x} \cdot e^x$$
$$= -2 \leftarrow \text{not zero}$$

Only get the uniq zero sol^n $\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

$\cosh x = \dfrac{e^x + e^{-x}}{2}$, $\sinh x = \dfrac{e^x - e^{-x}}{2}$

They form a <u>basis</u> too.

<u>EX:</u> $S' = \{ f \in C^0(\mathbb{R}) \text{ with } f(x) \geq 0, \forall x \}$

Is S' a vector <u>subspace</u> of $C^0(\mathbb{R})$?

<u>Subspace Test:</u> Only <u>two</u> things to check.

1) Closed under addition?

2) Closed under mult. by c? No!

Ex: $f(x) = x^2$, $c = -1$

$cf = -x^2 \leftarrow$ out of S

EX: M_{mn} = all $m \times n$ matrices

Basis? $M_{22} = \text{span}\{\underbrace{\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\1&0\end{bmatrix}, \begin{bmatrix}0&0\\0&1\end{bmatrix}}_{\text{basis}}\}$

M_{22} 4-dim$^{\ell}$

EX: P_n = polynomials of degree $\leq n$

$P_3 = \text{span}\{\underbrace{x^3, x^2, x, 1}_{\text{basis}}\}$

Indep. $\underbrace{c_1 x^3 + c_2 x^2 + c_3 x + c_4}_{P(x)} \equiv 0$

$$\frac{d^3 P}{dx^3} = 3!\, c_1 = 0, \quad \frac{d^2 P}{dx^2} = 2!\, c_2 = 0$$

etc.

Inner product spaces: $\vec{a} \cdot \vec{b}$

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

$$\vec{a}^T \vec{b} = [a_1 \ a_2 \cdots a_n] \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \text{same thing}$$

Properties of dot product: box on p. 312.

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

$$(\vec{a}_1 + \vec{a}_2) \cdot \vec{b} = \vec{a}_1 \cdot \vec{b} + \vec{a}_2 \cdot \vec{b}$$

$$\vec{a} \cdot \vec{a} \ \begin{cases} > 0 & \text{if } \vec{a} \neq \vec{0} \\ = 0 & \text{if } \vec{a} = \vec{0} \end{cases}$$

Norm: $\|\vec{a}\| = \sqrt{\vec{a} \cdot \vec{a}} = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$

$$\vec{a} \cdot \vec{b} = 0 \leftarrow \vec{a} \perp \vec{b}$$

Great thing: Orthogonal basis: $\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_n$

$$\vec{u}_i \cdot \vec{u}_j = 0 \text{ if } i \neq j.$$

$$\vec{u}_k \cdot \vec{b} = (c_1 \vec{u}_1 + \cdots + c_n \vec{u}_n) \cdot \vec{u}_k$$

$$= c_1 \underbrace{\vec{u}_1 \cdot \vec{u}_k}_{0} + \cdots + c_k \underbrace{\vec{u}_k \cdot \vec{u}_k}_{\text{not zero}} + \cdots + c_n \underbrace{\vec{u}_n \cdot \vec{u}_k}_{0}$$

Get $c_k = \dfrac{\vec{u}_k \cdot \vec{b}}{\vec{u}_k \cdot \vec{u}_k}$ easy!

Even easier if $\|\vec{u}_k\| = 1$, $\leftarrow$ Orthonormal

$\underline{\text{EX}}$: $C([0,\pi]) \leftarrow$ continuous fcns on $[0,\pi]$.

a vector space.

$f, g \in C([0,\pi])$.

$$f \cdot g = \underbrace{(f,g)}_{\text{notation}} \stackrel{\text{def}^n}{=} \int_0^\pi f(x)g(x)\,dx$$

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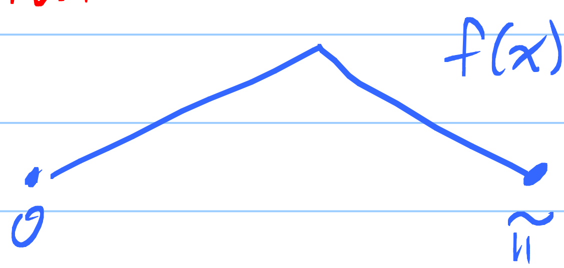


$$f(x) \stackrel{?}{=} \sum_{n=1}^{\infty} c_n \sin nx \qquad \text{Fourier Series}$$

$$\int_0^\pi \sin nx \sin mx\,dx = \begin{cases} \frac{\pi}{2} & n=m \\ 0 & n \neq m \end{cases}$$

$$\sin nx \perp \sin mx \quad n \neq m \;!$$

Hmmm. $\left(f = \sum c_n \sin nx \right) \cdot \sin mx$

$$(f, \sin mx) = c_m \underbrace{(\sin mx, \sin mx)}_{\pi/2}$$

$$\boxed{c_m = \frac{2}{\pi}\int_0^{\pi} f(x) \sin mx \, dx}$$

Hilbert Space: Infinite dimensional inner product space. $L^2([0,\pi]) = f : \int_0^{\pi} f(x)^2 dx < \infty$

(f,g) = inner product. $\{\sin nx\}_{n=1}^{\infty}$ orthog basis.

Linear Transformations: $T: \mathbb{R}^n \to \mathbb{R}^m$

Day one: Showed $T\vec{x} = \mathbb{A}\vec{x}$.

$$\vec{x} = x_1 \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \cdots + x_n \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}$$

$$T\vec{x} = x_1 T\vec{e}_1 + \cdots x_n T\vec{e}_n$$

$$= \underbrace{[T\vec{e}_1, \ldots, T\vec{e}_n]}_{\mathbb{A}\ m\times n} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$$(T_1 \circ T_2)(\vec{x}) = T_1(T_2\vec{x})$$
$$= T_1(\mathbb{A}_2\vec{x})$$
$$= \mathbb{A}_1(\mathbb{A}_2\vec{x})$$
$$= \underbrace{(\mathbb{A}_1\ \mathbb{A}_2)}_{\sim\, T_1 \circ T_2}\vec{x}$$

<u>EX</u>: $T : \mathbb{R}^3 \to \mathbb{R}^3$ $\qquad T = \mathbb{A}\vec{x}$ $\leftarrow$ 3×3

T maps <u>onto</u> $\mathbb{R}^3$ exactly when

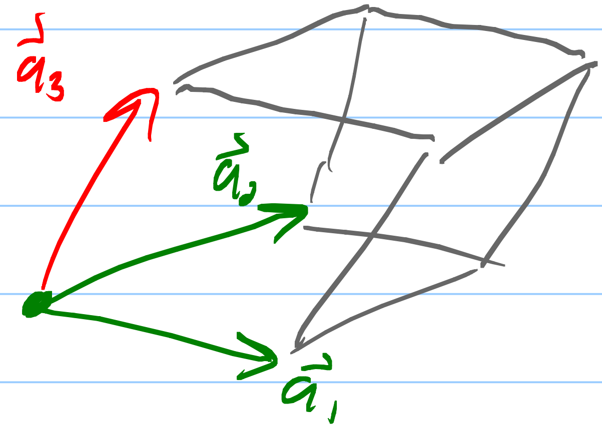
$$\left.\begin{array}{l} \mathbb{A} \text{ non-singular} \\ \det(\mathbb{A}) \neq 0 \\ \text{Rank}(\mathbb{A}) = 3 \\ \text{rows } \mathbb{A} \text{ indep} \end{array}\right\} \text{same!}$$

$T^{-1} = ?$

$$T\vec{x} = \vec{y}$$
$$\mathbb{A}\vec{x} = \vec{y}$$
$$\vec{x} = \mathbb{A}^{-1}\vec{y} = T^{-1}\vec{y}$$

Aha! $T^{-1} \sim \mathbb{A}^{-1}$.

Meaning of det(A): $[\vec{a}_1, \vec{a}_2, \vec{a}_3] = A$



$$\det(A) = \pm \text{Volume(box)}$$

$\det(A) \neq 0$ Solid box

$\det(A) = 0$ Smashed box (could be a line!)