Lesson 7 8.1 Eigenvalues, Eigenvectors

$y''-y=0$. Try $y=e^{\lambda t}$.

$$\lambda^2 e^{\lambda t} - e^{\lambda t} \overset{\text{want}}{=} 0$$

$$\underbrace{(\lambda^2-1)}_{=0} e^{\lambda t} = 0 \qquad \lambda = 1, -1$$

$$y = c_1 e^{t} + c_2 e^{-t}$$

System:
$$\begin{cases} \dfrac{dx_1}{dt} = -3x_1 + 2x_2 \\[2ex] \dfrac{dx_2}{dt} = x_1 - 2x_2 \end{cases}$$

Try $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} e^{\lambda t}$

$x_1 = a_1 e^{\lambda t}$

$x_2 = a_2 e^{\lambda t}$

$$\begin{cases} \lambda a_1 e^{\lambda t} \overset{\text{want}}{=} -3a_1 e^{\lambda t} + 2a_2 e^{\lambda t} \\ \lambda a_2 e^{\lambda t} = a_1 e^{\lambda t} - 2a_2 e^{\lambda t} \end{cases}$$

$$\begin{cases} 0 = (-3-\lambda)a_1 + 2a_2 \\ 0 = a_1 + (-2-\lambda)a_2 \end{cases}$$

$$\lambda \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{bmatrix} -3 & 2 \\ 1 & -2 \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

$$\begin{bmatrix} -3-\lambda & 2 \\ 1 & -2-\lambda \end{bmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \vec{0}$$

$$\boxed{\lambda \vec{a} = A \vec{a}}$$ ← want $\vec{a} \neq \vec{0}$

$$\underbrace{(A - \lambda I)}_{\text{must be bad!}} \vec{a} = \vec{0}$$

want $\vec{a} \neq \vec{0}$

λ is a <u>eigenvalue</u>, $\vec{a}$ <u>eigenvector</u> $(\vec{a} \neq 0)$

Need $$\boxed{\det(A - \lambda I) = 0}$$ ← Characteristic Eqn.

$$\det \begin{bmatrix} -3-\lambda & 2 \\ 1 & -2-\lambda \end{bmatrix} = 0$$

$$(-3-\lambda)(-2-\lambda) - 2 = 0$$

$$\lambda^2 + 5\lambda + 4 = 0$$

$$(\lambda+1)(\lambda+4) = 0 \qquad \boxed{\lambda = -1, -4} \leftarrow \text{e. vals.}$$

<u>Case $\lambda = -1$:</u> $\qquad (A - \underset{\lambda=-1}{\lambda} I)\vec{a} = 0$

$$\left[\begin{array}{cc|c} -3-(-1) & 2 & 0 \\ 1 & -2-(-1) & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} -2 & 2 & 0 \\ 1 & -1 & 0 \end{array}\right]$$

$$\rightsquigarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad a_1 = a_2 = s$$

a_1 bound $\qquad a_2$ free

Let $a_2 = s$

List: $\begin{cases} a_1 = s \\ a_2 = s \end{cases}$ $\qquad \vec{a} = s\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ <u>with $s \neq 0$</u>

Def^{n}: The eigenspace associated to an e-val λ is $E_\lambda = \text{null}(A - \lambda I)$ ← vector space!

= Set of all e-vects for λ plus the zero vect.

Above: $E_{\lambda=-1} = \text{Span}\left\{\begin{pmatrix}1\\1\end{pmatrix}\right\}$

For $\lambda = -4$:

$$\left[\begin{array}{cc|c} -3-(-4) & 2 & 0 \\ 1 & -2-(-4) & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 2 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \begin{aligned} a_1 &= -2a_2 \\ &= -2s \end{aligned}$$

a_1 bound $\qquad$ a_2 free

$a_2 = s$

$\vec{a} = \begin{pmatrix}-2s\\s\end{pmatrix} = s\begin{pmatrix}-2\\1\end{pmatrix}$ $\qquad E_{\lambda=-4} = \text{span}\left\{\begin{pmatrix}-2\\1\end{pmatrix}\right\}$

$$\vec{x} = c_1\begin{pmatrix}1\\1\end{pmatrix}e^{-t} + c_2\begin{pmatrix}-2\\1\end{pmatrix}e^{-4t}$$

EX: $A = \begin{bmatrix} -3 & -1 \\ 2 & -1 \end{bmatrix}$

Step 1: $\boxed{\det(\mathbb{A} - \lambda \mathbb{I}) = 0}$ ← e.val eqn

$$\det \begin{bmatrix} -3-\lambda & -1 \\ 2 & -1-\lambda \end{bmatrix} = 0$$

$$(-3-\lambda)(-1-\lambda) + 2 = 0$$

$$\lambda^2 + 4\lambda + 5 = 0$$

ouch! $\lambda = -2 \pm i$ (quad. form.)

Step 2: $\boxed{(\mathbb{A} - \lambda \mathbb{I})\vec{a} = \vec{0}}$ ← e.vect system

$\lambda = -2+i$

For $\lambda = -2+i$

$$\left[\begin{array}{cc|c} -3-(-2+i) & -1 & 0 \\ 2 & -1-(-2+i) & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} -1-i & -1 & 0 \\ 2 & 1-i & 0 \end{array}\right]$$

$R_2 = (-1+i)R_1$

$$\rightsquigarrow \left[\begin{array}{cc|c} -1-i & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

a_1 bound

a_2 free

$a_2 = s$

$a_1 = \dfrac{a_2}{(-1-i)}$

$a_1 = s\left(-\frac{1}{2} + \frac{i}{2}\right)$

$$\begin{cases} a_1 = s\left(-\frac{1}{2}+\frac{i}{2}\right) \\ a_2 = s \end{cases} \qquad \vec{a} = s\begin{pmatrix} -\frac{1}{2}+\frac{1}{2}i \\ 1 \end{pmatrix}$$

complex e.vect.

<u>For $\lambda = -2-i$</u>: ← conjugate of first e.val.

$\vec{a}$ = conjugate of above = $\begin{pmatrix} -1/2-\frac{1}{2}i \\ 1 \end{pmatrix}$

Why: Take conjugate of $\mathbb{A}\vec{a} = \lambda\vec{a}$.

<u>Ex</u>: $\mathbb{A} = \begin{bmatrix} -3 & 1 \\ -1 & -1 \end{bmatrix}$ $\quad \det\begin{pmatrix} -3-\lambda & 1 \\ -1 & -1-\lambda \end{pmatrix} =$

$\lambda^2 + 4\lambda + 4 =$

Ouch! $\lambda = -2$ is a <u>double</u> root. $\quad (\lambda+2)^2 = 0$

<u>Word</u>: $\lambda = -2$ is an eigenvalue of "algebraic multiplicity 2."

<u>For $\lambda = -2$</u>: $\quad (\mathbb{A} - \lambda\mathbb{I})\vec{a} = 0$

$\lambda = -2$

$$\left[\begin{array}{cc|c} -3-(-2) & 1 & 0 \\ -1 & -1-(-2) & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} -1 & 1 & 0 \\ -1 & 1 & 0 \end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Only get a <u>one</u> dim'l e-space. Darn!

$$E_{\lambda=-2} = \text{span}\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}.$$

<u>Word</u>: Dimension of eigenspace for e-val λ is the "geometric multiplicity of λ."

<u>Note</u>: geom. mult. $\leq$ alg. mult. $\leq n$

$<$ bad.

$=$ good.

Matrix defective

Defect = (alg. mult.) − (geom. mult.)

<u>Anything can happen</u>!

$$\mathbb{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbb{B} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbb{C} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\det(\text{Matrix} - \lambda \mathbb{I}) = (1-\lambda)^3$$

$\lambda = 1$ e.val of algebraic mult. 3.

Case A: $(A - \underset{\lambda=1}{\lambda} I)\vec{a} = \vec{0}$

$$\begin{bmatrix} 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \\ 0 & 0 & 0 & | & 0 \end{bmatrix}$$

free!

$$\vec{a} = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} = s_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + s_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$E_{\lambda=1} = \text{span}\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ ← 3 dim^l

geom. mult. $= 3$

Case B: $(A - \underset{\lambda=1}{\lambda} I)\vec{a} = 0$

$a_2 = 0$ →

$$\begin{array}{cccccc} & & & & & \\ \left[\begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}\right. & \left|\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right] \\ \begin{matrix} a_1 & a_2 & a_3 \\ \text{free} & \text{bound} & \text{free} \end{matrix} & \end{array}$$

$a_1 = s_1$

$a_3 = s_2$

$$\vec{a} = \begin{pmatrix} s_1 \\ 0 \\ s_2 \end{pmatrix} = s_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + s_2 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$E_{\lambda=1} = \text{Span}\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\end{pmatrix} \right\} \leftarrow 2 \text{ dim}^{l}$$

geom. mult. = 2

Case C: $(\mathbb{A} - \underbrace{\lambda}_{\lambda=1} \mathbb{I})\vec{a} = \vec{0}$

$$\begin{array}{cccccl} & \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] & \begin{array}{l} \leftarrow a_2 = 0 \\ \leftarrow a_3 = 0 \\ \end{array} \\ & \begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ a_1 & a_2 & a_3 \\ \text{free} & \text{bound} & \text{bound} \end{array} & \end{array}$$

$a_1 = s$

$$\vec{a} = \begin{pmatrix}s\\0\\0\end{pmatrix} = s\begin{pmatrix}1\\0\\0\end{pmatrix}$$

$$E_{\lambda=1} = \text{Span}\left\{ \begin{pmatrix}1\\0\\0\end{pmatrix} \right\} \leftarrow 1 \text{ dim}^{l}$$

geom mult = 1. Defect = 3-1 = 2.