Lesson 8   8.3, 8.5   Lessons 7, 8 due Wed.

Remarks: 1) $(\underset{2\times 2}{A} - \underset{\text{e-val}}{\lambda} I)\vec{a} = \vec{0} \rightsquigarrow \left[\begin{array}{cc|c} a & b & 0 \\ 0 & 0 & 0 \end{array}\right]$

Aha! See e-vect $\begin{pmatrix} b \\ -a \end{pmatrix}$ or $\begin{pmatrix} -b \\ a \end{pmatrix}$.

2) Say $\begin{pmatrix} 7/13 \\ -2/13 \end{pmatrix}$ is an e-vect.

Nicer one: $\begin{pmatrix} 7 \\ -2 \end{pmatrix}$

$\mathbb{R}$, A $n\times n$

Symmetric: $A^T = A$

$$\begin{bmatrix} e & 13 \\ 13 & \tilde{\pi} \end{bmatrix}$$

Skew-symmetric: $A^T = -A$

$$\begin{bmatrix} 0 & -13 \\ 13 & 0 \end{bmatrix}$$

Orthogonal: $A^T = A^{-1}$

$$\begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$$

$$\vec{u}\cdot\vec{v} = \sum_{i=1}^{n} u_i v_i$$

$\mathbb{C}$

Hermitian: $\overline{A}^T = A$

$$\begin{bmatrix} e & 1-i \\ 1+i & \tilde{\pi} \end{bmatrix}$$

Skew Hermitian: $\overline{A}^T = -A$

$$\begin{bmatrix} i & -1+i \\ 1+i & 2i \end{bmatrix}$$

Unitary: $\overline{A}^T = A^{-1}$

$$\begin{bmatrix} \frac{1}{2}i & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2}i \end{bmatrix}$$

$$\boxed{\vec{u}\cdot\vec{v} = \sum_{i=1}^{n} u_i \overline{v_i}}$$

complex inner prod

Facts: 1) A symmetric $\Rightarrow$ A has real e-vals.

E-vects for different e-vals are $\perp$.

2) A skew-symm $\Rightarrow$ e-vals pure imaginary

3) A orthog $\Leftrightarrow$ rows of A orthonormal

$\Leftrightarrow$ cols of A orthonormal

In this case, rows (or cols) form a basis for $\mathbb{R}^n$.

4) A orthog. Then $\det(A) = \pm 1$.

Why 3: $A^T = A^{-1}$

col j = row j of A

$$\text{row } i \; \underbrace{\begin{bmatrix} & & \\ \hline & & \\ & & \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} & \vert & \\ & \vert & \\ & \vert & \end{bmatrix}}_{A^{-1} = A^T} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

length row i = 1

$$(\text{row } i)\cdot(\text{row } j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

rows $\perp$

Why 4: Big Fact $\det(AB) = \det(A)\det(B)$

A orthog: $A^T = A^{-1}$

$$A \underbrace{A^{-1}}_{A^T} = I$$

$$\det(AA^T) = \det(I) = 1$$

$$\det(A)\det(A^T) = 1$$

But $\det(A^T) = \det(A)$. $\det(A)^2 = 1$ ✓

<u>Notation</u>: $(\vec{u}, \vec{v}) = \vec{u} \cdot \vec{v} = [u_1 \cdots u_n] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \vec{u}^T \vec{v}$

<u>Fact</u>: $\boxed{(A\vec{u}, \vec{v}) = (\vec{u}, A^T \vec{v})}$

<u>Why</u>: $(A\vec{u}, \vec{v}) = (A\vec{u})^T \vec{v}$

$= (\vec{u}^T A^T) \vec{v}$

$= \vec{u}^T (A^T \vec{v})$

$= (\vec{u}, A^T \vec{v})$ ✓

$(AB)^T = B^T A^T$

Why are e-vects for different e-vals of a symmetric matrix $\perp$?

λ's diff. $\lambda_1 \vec{a}_1 = A\vec{a}_1$

$\lambda_2 \vec{a}_2 = A\vec{a}_2$

$\lambda_1 (\vec{a}_1, \vec{a}_2) = (\lambda_1 \vec{a}_1, \vec{a}_2) = (A\vec{a}_1, \vec{a}_2)$

$= (\vec{a}_1, \underbrace{A^T}_{=A} \vec{a}_2) = (\vec{a}_1, A\vec{a}_2)$

$= (\vec{a}_1, \lambda_2 \vec{a}_2) = \lambda_2 (\vec{a}_1, \vec{a}_2)$

$\underbrace{(\lambda_1 - \lambda_2)}_{\neq 0} (\vec{a}_1, \vec{a}_2) = 0$

must $= 0$.

Complex case: $\mathbb{A}$ hermitian, $\lambda \vec{a} = \mathbb{A}\vec{a}$

Big Fact: $(\mathbb{A}\vec{u}, \vec{v}) = (\vec{u}, \overline{\mathbb{A}}^T \vec{v})$

(complex inner prod)

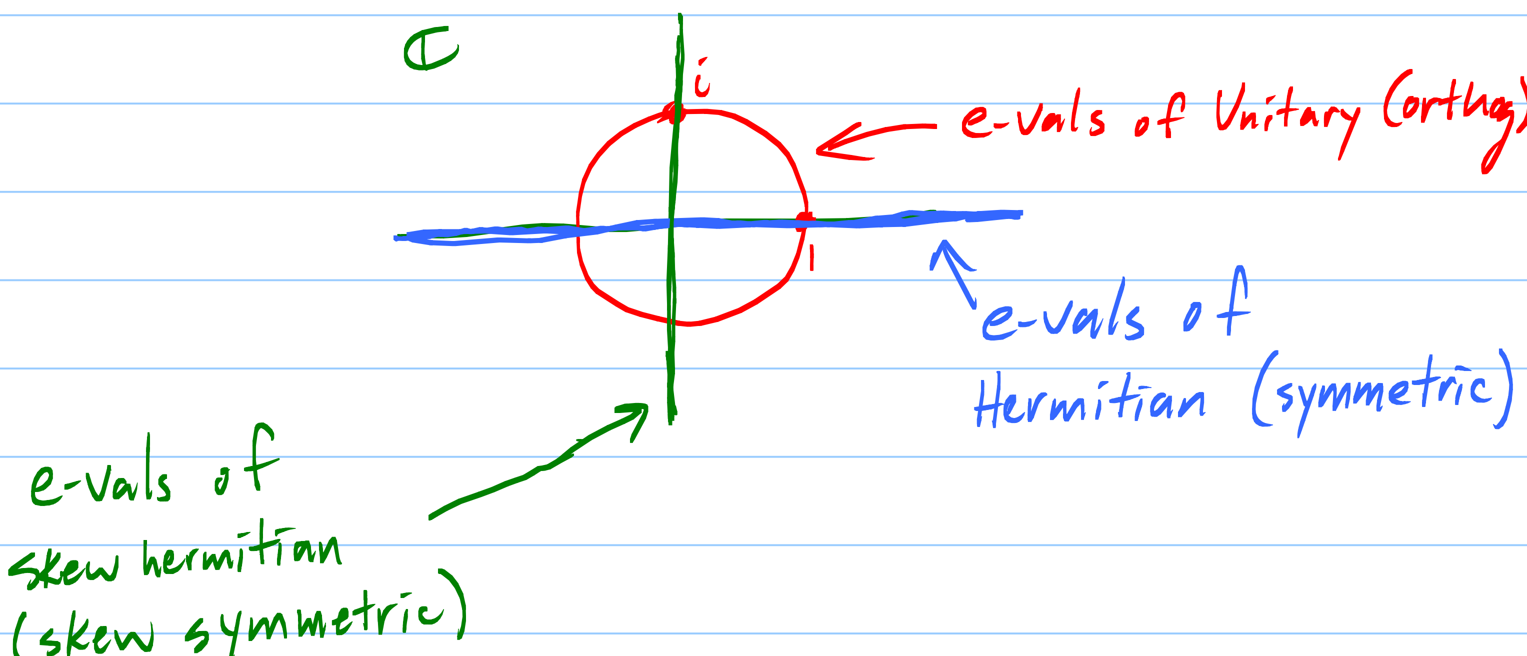
$$\lambda \underbrace{(\vec{a}, \vec{a})}_{\text{not zero}} = \cdots = (\vec{a}, \lambda \vec{a}) = \bar{\lambda} (\vec{a}, \vec{a})$$

Conclude that $\lambda = \bar{\lambda}$, i.e., that λ is real.

Note: Symmetric matrices are hermitian!

So symm matrices have only real e-vals.

Word: The spectrum of a matrix is the set of all e-vals (sol^{n}s to $\det(\mathbb{A} - \lambda \mathbb{I}) = 0$)



Fact: Orthogonal matrices preserve length and

Inner product, meaning

$$\|\mathbb{O}\vec{u}\| = \|\vec{u}\|$$

$$(\mathbb{O}\vec{u}, \mathbb{O}\vec{v}) = (\vec{u}, \vec{v})$$

Why: $(\mathbb{O}\vec{u}, \mathbb{O}\vec{v}) = (\vec{u}, \underbrace{\mathbb{O}^T}_{\mathbb{O}^{-1}}\mathbb{O}\vec{v})$

$$= (\vec{u}, \mathbb{I}\vec{v}) = (\vec{u}, \vec{v}) \checkmark$$

Take $\vec{u} = \vec{v}$: $\|\mathbb{O}\vec{u}\|^2 = \|\vec{u}\|^2$ $\checkmark$

p. 329: 24. $\mathbb{A}^{-1}$ exists $\iff$ e-vals λ_j are all <u>non-zero</u>. Then $\mathbb{A}^{-1}$ has e-vals $1/\lambda_j$.

λ e-val: $\det(\mathbb{A} - \lambda\mathbb{I}) = 0$

what if $\lambda = 0$?

<u>Facts</u>: $\mathbb{A}^{-1}$ exists $\iff \text{Rank}(\mathbb{A}) = n$

($\mathbb{A}$ $n \times n$) $\iff \det(\mathbb{A}) \neq 0$.

e-vals of $\mathbb{A}^{-1}$? λ e-val of $\mathbb{A}$: $\lambda\vec{a} = \mathbb{A}\vec{a}$

<u>Idea</u>: Hit it with $\mathbb{A}^{-1}$ on <u>left</u>. Pull out λ...

Fact: $\mathbb{A}\mathbb{A}^{-1} = \mathbb{I}$

$$\det(\mathbb{A}\mathbb{A}^{-1}) = \det(\mathbb{I})$$

$$\det(\mathbb{A})\det(\mathbb{A}^{-1}) = 1$$

$$\det(\mathbb{A}^{-1}) = [\det(\mathbb{A})]^{-1}$$