Lesson 10 4.1, 4.2 9,10,11 due Wed.

$$\hat{A} = P^{-1} A P$$

$$\underbrace{P}_{Q^{-1}} \hat{A} \underbrace{P^{-1}}_{Q} = A$$

Similarity is an equivalence relation: Reflexive, trasitive, etc.

Linear const. coeff ODE:

$$L[y] = a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$$

homog.

Sol^n in $C(\mathbb{R})$ ← vector space

Subspace Test: 1) y_1, y_2 sol^{n}s. Is $y_1 + y_2$?

$L[y_1] = 0$, $L[y_2] = 0$ $\quad$ $L[y_1 + y_2] =$

(Superposition Principle) $\quad$ $L[y_1] + L[y_2] = 0 + 0$ ✓

2) y is a sol^n. Is cy?

$L[y] = 0$ $\quad$ $L[cy] = cL[y] = c \cdot 0 = 0$ ✓

How to solve: Try $y = e^{rt}$:

$$a_n r^n e^{rt} + \cdots + a_1 r e^{rt} + a_0 e^{rt} \overset{\text{want}}{=} 0$$

$$\underbrace{(a_n r^n + \cdots + a_0)}_{\text{need } = 0} e^{rt} = 0$$

never 0.

Characteristic equation: $a_n r^n + \cdots + a_0 = 0$

Case 1: Real root r of mult. 1.

Get $y = e^{rt}$ as sol^n.

Case 2: Real root r of mult. $m > 1$.

Get sol^{n}s $e^{rt}, te^{rt}, \ldots, t^{m-1}e^{rt}$

Case 3: $r = a \pm bi$ complex roots. Get two real sol^{n}s $e^{at}\cos bt,\ e^{at}\sin bt$.

Why: $e^{(a+bi)t}$ is a complex sol^n.

$$e^{at} \cdot e^{bit} = e^{at}(\underbrace{\cos bt + i\sin bt}_{\text{Euler}})$$

$$= (\underbrace{e^{at}\cos bt}_{u}) + i\,(\underbrace{e^{at}\sin bt}_{v})$$

$$0 = L[u+iv] = \underbrace{L[u]}_{0} + i\,\underbrace{L[v]}_{0}$$

Get two real sol^{n}s from one complex

EX: $y^{(4)} + y'' = 0 \qquad r^4 + r^2 = 0 \qquad r^2(r^2+1) = 0$

$\underbrace{0, 0}_{e^{0t},\ te^{0t}},\ \pm i \quad e^{0t}\cos t,\ e^{0t}\sin t$

$1, t, \cos t, \sin t$

$y = c_1 + c_2 t + c_3 \cos t + c_4 \sin t$

Case 4: $r = a \pm bi$ repeated m times

e.g. $(r^2+1)^5$, $\pm i \leftarrow$ mult 5

$$\begin{array}{l} 2m \\ \text{sol}^{\text{n}}\text{s} \end{array} \begin{cases} e^{at}\cos bt, \; te^{at}\cos bt, \; \ldots, \; t^{m-1}e^{at}\cos bt \\ e^{at}\sin bt, \; te^{at}\sin bt, \; \ldots, \; t^{m-1}e^{at}\sin bt \end{cases}$$

EX: $y^{(4)} + 2y'' + y = 0 \qquad (r^2+1)^2 = 0$

$y = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t$

Q: How do we know this is the gen$^{\text{l}}$ sol$^{\text{n}}$?

E. & U. Thm. $\quad a_n y^{(n)} + \cdots + a_0 y = 0$

Initial Value Problem

IVP

$$\begin{cases} y(0) = A_0 \\ y'(0) = A_1 \\ \vdots \\ y^{(n-1)}(0) = A_{n-1} \end{cases}$$

The sol$^{\text{n}}$ exists and it is unique.

$$\begin{cases} y(0) = c_1 y_1(0) + c_2 y_2(0) + \cdots + c_n y_n(0) \overset{\text{want}}{=} A_0 \\ y'(0) = c_1 y_1'(0) + \cdots + c_n y_n'(0) = A_1 \\ \vdots \\ y^{(n-1)}(0) = c_1 y_1^{(n-1)}(0) + \cdots + c_n y_n^{(n-1)}(0) = A_n \end{cases}$$

$$\underbrace{\begin{bmatrix} y_1(0) & \cdots & y_n(0) \\ y_1'(0) & \cdots & y_n'(0) \\ \vdots & & \\ y_1^{(n-1)}(0) & \cdots & y_n^{(n-1)}(0) \end{bmatrix}}_{\text{need det} \neq 0} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} A_0 \\ A_1 \\ \vdots \\ A_n \end{pmatrix}$$

anything

Wronskian $= W[y_1, \ldots, y_n] = W = \det \left[\begin{matrix} t\text{'s where} \\ 0\text{'s are} \end{matrix} \right]$

Fact: If $W(0) \neq 0$, then we can solve any and all IVP's. E & U Thm $\Rightarrow$ we have gen^l sol^n.

Systems of ODE's: $\vec{x}' = A\vec{x}$, $\quad \vec{x}(0) = \vec{x}_0$

$$\begin{cases} \dfrac{dx_1}{dt} = 3x_1 + 5x_2 \\ \dfrac{dx_2}{dt} = 2x_1 - x_2 \end{cases} \qquad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \qquad \vec{x}' = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$$

Try $\vec{x} = \vec{a}\, e^{\lambda t}$. Found we need

$$\det(\mathbb{A} - \lambda \mathbb{I}) = 0 \leftarrow \lambda = \text{e.val for } \mathbb{A}$$

$$(\mathbb{A} - \lambda \mathbb{I})\vec{a} = \vec{0} \leftarrow \vec{a} = \text{e.vect for } \lambda$$

Case $\mathbb{A}$ real symmetric: Then $\mathbb{A}$ has

real e-vals $\lambda_1, \lambda_2, \ldots, \lambda_n$ ← some may be repeated.

and a full set of e-vects. $\underbrace{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n}_{\text{lin. indep. (basis)}}$

$$\vec{x} = c_1 \vec{a}_1 e^{\lambda_1 t} + \cdots + c_n \vec{a}_n e^{\lambda_n t}$$

Q: Is this the gen^{l} sol^{n}?

Equiv Q: Can I solve any and all IVP's?

$$\vec{x}(0) = c_1 \vec{a}_1 e^{\lambda_1 0} + \cdots + c_n \vec{a}_n e^{\lambda_n 0} \overset{\text{want}}{=} \underbrace{\vec{x}_0}_{\text{anything}}$$

$$\underbrace{\begin{bmatrix} \vec{a}_1, & \ldots, & \vec{a}_n \end{bmatrix}}_{\text{need } \det \neq 0} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} = \vec{x}_0$$

Yes! Because $\vec{a}_j$'s indep!

Def^n: Wronskian $= \det\left[\vec{x}_1, \ldots, \vec{x}_n\right]$

$$= \det\left[\vec{a}_1 e^{\lambda_1 t}, \ldots, \vec{a}_n e^{\lambda_n t}\right]$$

$$= e^{(\lambda_1 + \cdots + \lambda_n)t} \underbrace{\det\left[\vec{a}_1, \ldots, \vec{a}_n\right]}_{\substack{\neq 0 \text{ number} \\ = W(0).}}$$

Never zero!

Fact: If $W(0) \neq 0$, we have the gen^l sol^n.

Can turn one high order ODE into a big first order system.

EX:
$$3y''' + 2\underset{\substack{\uparrow \\ x_3}}{y''} + \underset{\substack{\uparrow \\ x_2}}{y'} + \underset{\substack{\uparrow \\ x_1}}{y} = 0 \qquad (*)$$

Let $x_1 = y$, $x_2 = y'$, $x_3 = y''$

$$\begin{cases} x_1' = y' = x_2 \\ x_2' = y'' = x_3 \\ x_3' = y''' = \underbrace{-\frac{2}{3}x_3 - \frac{1}{3}x_2 - \frac{1}{3}x_1}_{?} \end{cases}$$

Key: Get $\underbrace{\qquad}_{?}$ from $(*)$

Solve $(*)$ for $y''' = -\frac{2}{3}\underset{x_3}{y''} - \frac{1}{3}\underset{x_2}{y'} - \frac{1}{3}\underset{x_1}{y}$

$$\vec{x}\,' = \begin{pmatrix} x_2 \\ x_3 \\ -\frac{1}{3}x_1 - \frac{1}{3}x_2 - \frac{2}{3}x_3 \end{pmatrix}$$

$$= \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\frac{1}{3} & -\frac{1}{3} & -\frac{2}{3} \end{bmatrix}}_{\mathbb{A}} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\vec{x}}$$

$$\det \begin{bmatrix} a-\lambda & k & k \\ k & a-\lambda & k \\ k & k & a-\lambda \end{bmatrix} = ?$$

kill

$$= \det \begin{bmatrix} a-\lambda-k & k & k \\ k-a+\lambda & a-\lambda & k \\ 0 & k & a-\lambda \end{bmatrix} R_2 + R_1$$

kill

$= (a-\lambda-k)\ \det(2\times 2 \text{ matrix})$

factored out one root!