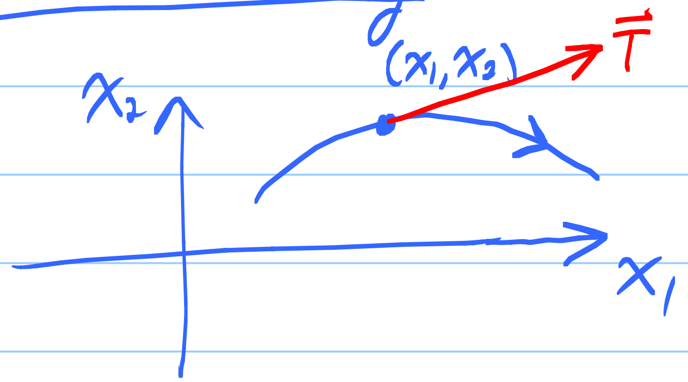
Lesson 11 4.3

$$\begin{cases} \dfrac{dx_1}{dt} = F_1(x_1, x_2) \\ \dfrac{dx_2}{dt} = F_2(x_1, x_2) \end{cases} \qquad \vec{x}(0) = \vec{x}_0$$

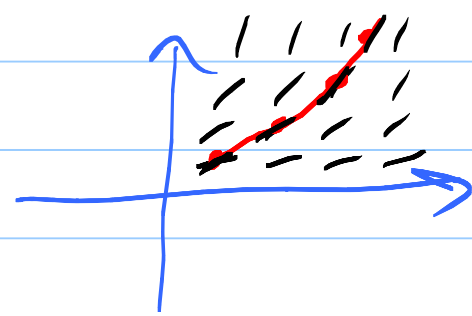
no t on right
"autonomous"

Geometric meaning: Sol^n $\vec{x} = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$



$$\vec{T} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} F_1(x_1,x_2) \\ F_2(x_1,x_2) \end{pmatrix}$$

Direction Field:



on grid, at (x_1, x_2) draw arrow pointing in $\begin{pmatrix} F_1(x_1,x_2) \\ F_2(x_1,x_2) \end{pmatrix}$ dir

http://math.rice.edu/~dfield/dfpp.html
pplane

Linear case: $\begin{cases} \dfrac{dx_1}{dt} = 5x_1 - x_2 \\ \dfrac{dx_2}{dt} = 3x_1 + x_2 \end{cases}$ $\vec{x}\,' = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \vec{x}$

e-vals $\lambda = 2, 4$

e-vects $\begin{pmatrix}3\\1\end{pmatrix}$ $\begin{pmatrix}1\\1\end{pmatrix}$

$$\vec{x} = \underline{\underline{c_1}}\begin{pmatrix}3\\1\end{pmatrix}e^{2t} + \underline{\underline{c_2}}\begin{pmatrix}1\\1\end{pmatrix}e^{4t}$$

t big : $\vec{x}$ way out

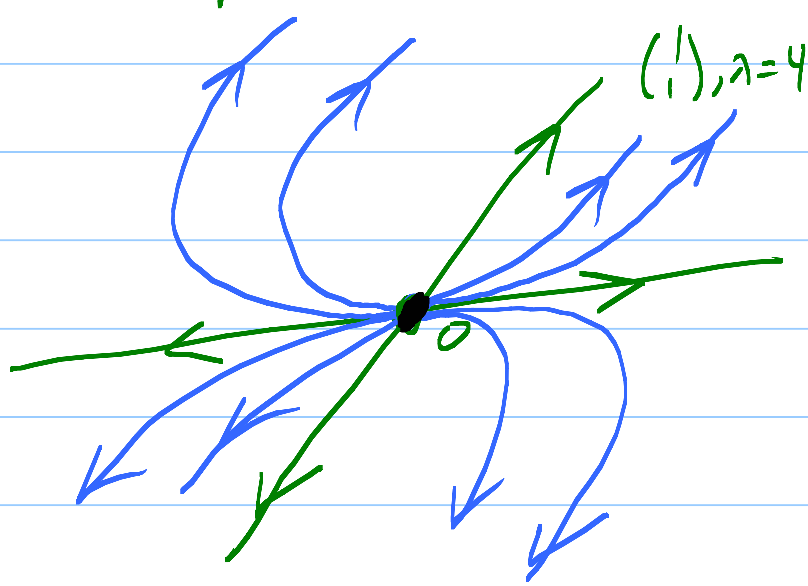
$t \to -\infty$: near origin

λ closest to zero

$$\vec{x} = e^{2t}\left[\underbrace{c_1\begin{pmatrix}3\\1\end{pmatrix}}_{\text{big}} + c_2\underbrace{\begin{pmatrix}1\\1\end{pmatrix}e^{2t}}_{t\to-\infty \text{ small}}\right]$$

Near origin, sol^{n}s hug e-vect cor. to $\lambda = 2$.

Step 1: Graph easy sol^{n}s: $c_1 = \pm 1$, $c_2 = 0$
$c_2 = \pm 1$, $c_1 = 0$



$\begin{pmatrix}3\\1\end{pmatrix}, \lambda = 2$

$c_1 = 1$
$c_2 = 0$

$c_1 = -1$
$c_2 = 0$

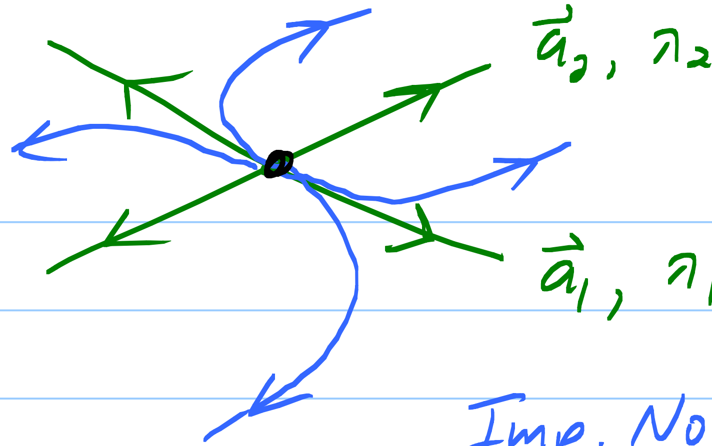
Origin: Type: Improper node, Unstable

Method: $\vec{x}' = A\vec{x}$. Origin is a critical pt.

Get e-vals λ_1, λ_2 and e-vects.

Case 1: λ_1, λ_2 real same sign

a) $0 < \lambda_1 < \lambda_2$.

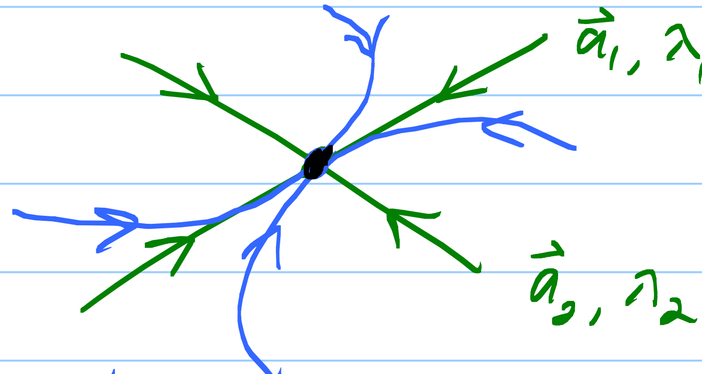


$\leftarrow$ λ_1 closest to zero

Imp. Node, Unstable

b) $\lambda_2 < \lambda_1 < 0$

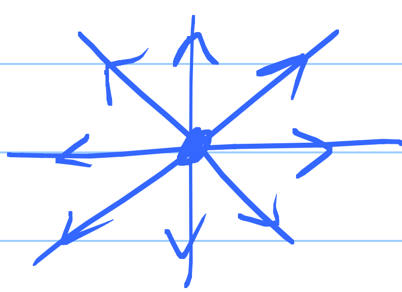
$\uparrow$ closest to zero



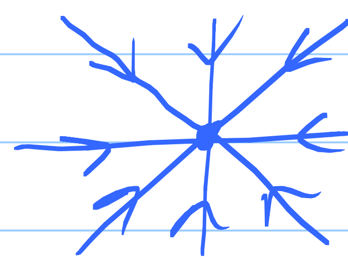
Improper Node, Asymptotically Stable.

Node Rule: Near the origin, the trajectories hug $\pm$ the e-vect corresponding to the e-val closest to zero. (Far away, the trajectories look parallel to the other e-vect.)

c) $\lambda_1 = \lambda_2$ real, $\neq 0$.
Full set of e-vects.



$\lambda_1 > 0$ Unstable



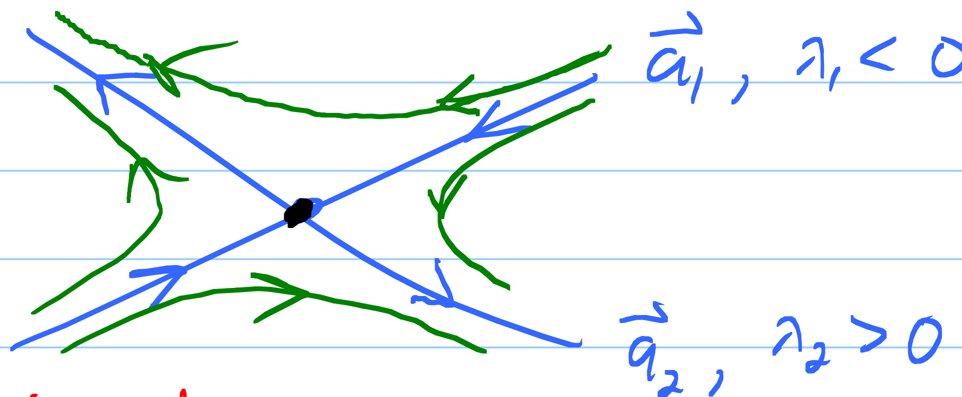
$\lambda_1 < 0$ A. Stable

Ex: $\vec{x}' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \vec{x}$ $\quad \vec{x} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^t$

$$= \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} e^t$$

Subcase: $\lambda_1 = \lambda_2$, but not a full set of e-vects.

Case $\lambda_1 < 0 < \lambda_2$: (real)



$t \to -\infty$: big, small

$$\vec{x} = c_1 \vec{a}_1 e^{\lambda_1 t} + c_2 \vec{a}_2 e^{\lambda_2 t}$$

$t \to \infty$: small, big

Saddle Point. Unstable

Case $\lambda = a \pm bi$: Complex.

For $\lambda = a + bi$, get complex e-vect $\vec{A} + i\vec{B}$.

$$\vec{x} = (\vec{A} + \vec{B}i) e^{(a+bi)t} = (\vec{A} + i\vec{B})(e^{at}\cos bt + i e^{at}\sin bt)$$

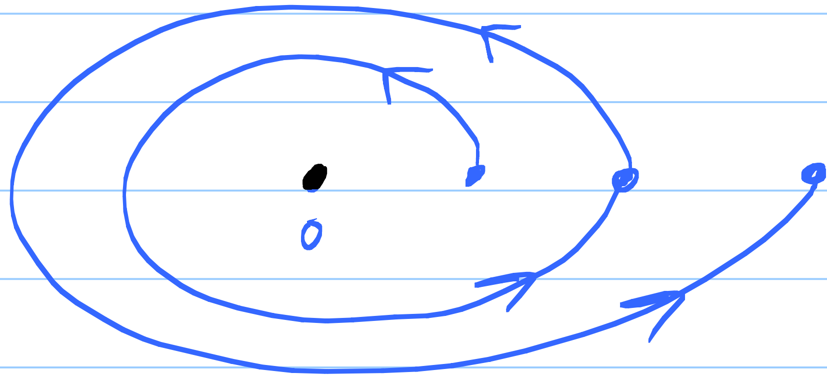
$$= \underbrace{\left(\vec{A}e^{at}\cos bt - \vec{B}e^{at}\sin bt\right)}_{\text{real sol}^n} + i\left(\underbrace{\vec{A}e^{at}\sin bt + \vec{B}e^{at}\cos bt}_{\text{real sol}^n}\right)$$

$i^2 = -1$

two real sol^{n}s from one complex sol^n!

$$\vec{x} = c_1 e^{at}\left(\vec{A}\cos bt - \vec{B}\sin bt\right) + c_2 e^{at}\left(\vec{A}\sin bt + \vec{B}\cos bt\right)$$

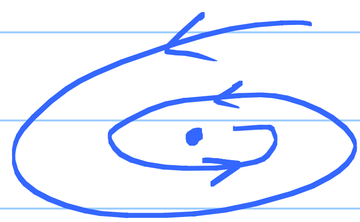
Subcase: $a > 0$; e^{at} gets bigger, bigger. Rest is periodic.



Spiral (out)

Unstable

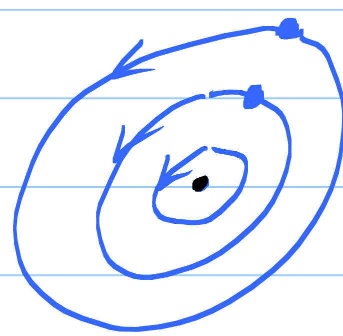
Subcase: $a < 0$: e^{at} shrinks.



Spiral (in),

A. Stable

Subcase: $a = 0$:



← ellipses

Type: Center

Stable (but not asymptotically stable)

Clockwise or Counterclockwise?
CW CCW

EX: $\vec{x}' = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \vec{x}$ $\quad \lambda = \underbrace{-1}_{a=-1} \pm i$

Spiral (in), A. stable.

Test the field at $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$:

At $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\vec{x}' = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$

Aha! Clockwise