Lesson 12   9, 10, 11 due Wed., 12, 13, 14 due next Wed.

Exam 1 in class, Fri., Sept. 27

Closed books, notes, no calculators, etc.

One crib sheet, hand written, both sides.

<u>EX:</u> $\vec{x}\,' = \begin{bmatrix} -2 & 2 \\ -2 & -2 \end{bmatrix} \vec{x}$ $(-2-\lambda)^2 + 4 = 0$

$\lambda = -2 \pm 2i$

<u>For $\lambda = -2+2i$:</u> $(A - \lambda I)\vec{a} = \vec{0}$

$\lambda = -2+2i$

$$\left[\begin{array}{cc|c} -2-(-2+2i) & 2 & 0 \\ -2 & -2-(-2+2i) & 0 \end{array}\right] \sim \begin{array}{cc} A & B \end{array} \left[\begin{array}{cc|c} -2i & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{cc|c} A & B & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \begin{pmatrix} -B \\ A \end{pmatrix}, \begin{pmatrix} B \\ -A \end{pmatrix} \text{ are e-vects.}$$

$$\begin{pmatrix} 2 \\ 2i \end{pmatrix} \qquad \begin{pmatrix} B \\ -A \end{pmatrix} \qquad \text{Better: } \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Complex Sol^n $= \vec{a} e^{\lambda t} = \vec{a}\, e^{(-2+2i)t} = \vec{a}\, e^{-2t} e^{2it}$

$$\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] \left(e^{-2t}\cos 2t + i e^{-2t} \sin 2t\right) =$$

$$\left(\underline{\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2t}\cos 2t} \underset{i^2=-1}{-} \underline{\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t} \sin 2t} \right) + i \left(\underline{\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-2t} \sin 2t + \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-2t} \cos 2t} \right)$$

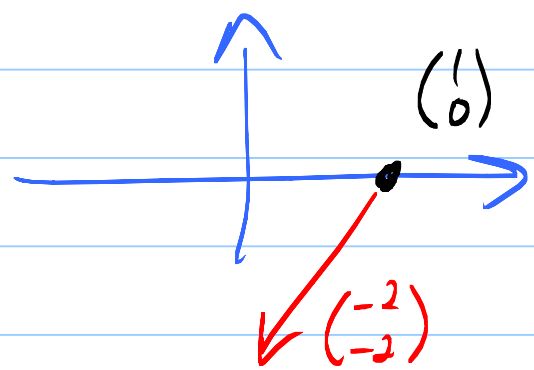
$= \vec{x}_1 + i\,\vec{x}_2 \leftarrow$ Get 2 real sol^{n}s.

Gen^l Sol^n $\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 = c_1 e^{-2t}\begin{pmatrix}\cos 2t\\ -\sin 2t\end{pmatrix} + c_2 e^{-2t}\begin{pmatrix}\sin 2t\\ \cos 2t\end{pmatrix}$

Origin: Spiral (in), Asympt. Stable (attracting).

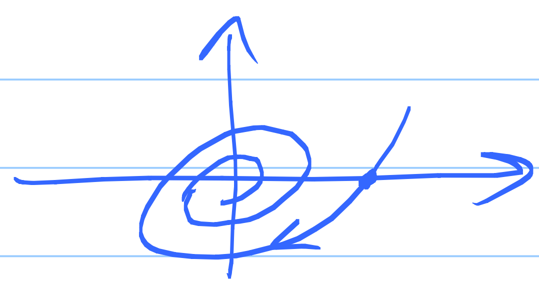
CW or CCW? Test field at $\begin{pmatrix}1\\0\end{pmatrix}$:

$\vec{x}\,'$ at $\begin{pmatrix}1\\0\end{pmatrix} = A\begin{pmatrix}1\\0\end{pmatrix} = \begin{bmatrix}-2 & 2\\ -2 & -2\end{bmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-2\\-2\end{pmatrix}$



At $\begin{pmatrix}s\\0\end{pmatrix}$: $\vec{x}\,' = \begin{pmatrix}-2s\\-2s\end{pmatrix}$

Clockwise.



Last case: λ repeated e-val, but only one e-vect.
(A defective: geom mult < alg. mult.).

Get one sol^n $\vec{x}_1 = \vec{a}\,e^{\lambda t}$ from one e-vect. $\vec{a}$.

Need $\vec{x}_2$. Try $\vec{x}_2 = \vec{a}\,t e^{\lambda t}$. Nope!

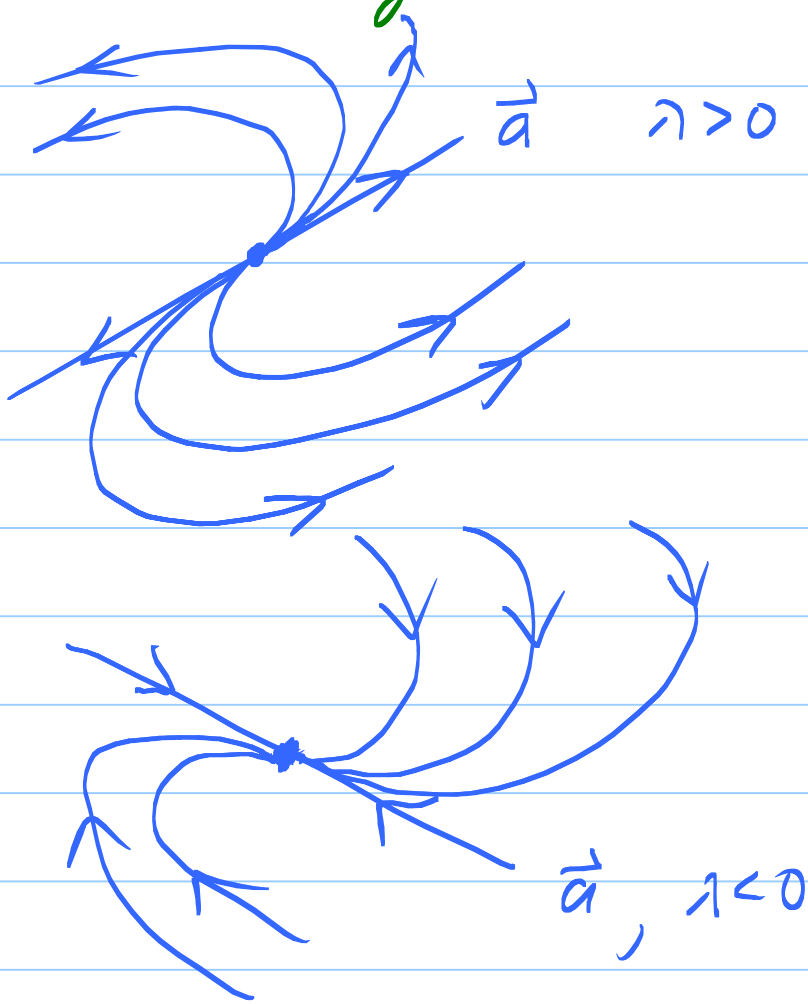
Better try: $\vec{x}_2 = \vec{a}\, t e^{\lambda t} + \vec{b} e^{\lambda t}$

$\vec{a}$ = e-vect.

$\boxed{(A - \lambda I)\vec{b} = \vec{a}}$

det = 0

Fact: Even though det = 0, this system has sol^{n}s $\vec{b}$. In fact, there are ∞-many sol^{n}s. Just pick one to get $\vec{x}_2$.



Improper Node, Unstable.

Improper Node, Asympt. Stabe (attracting)

Non-linear systems: How to linearize a C^2-smooth function.

$f(x) \cong f(x_0) + f'(x_0)(x - x_0)$ near x_0.

$$u(x,y) = u(x_0,y_0) + \frac{\partial u}{\partial x}(x_0,y_0)(x-x_0) + \frac{\partial u}{\partial y}(x_0,y_0)(y-y_0)$$
$$+ R(x,y) \leftarrow \text{super small near } (x_0,y_0)$$

where
$$\lim_{(x,y)\to(x_0,y_0)} \frac{R(x,y)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0$$

Taylor's Thm.

<u>Non-linear system</u>:
$$\begin{cases} \dfrac{dx}{dt} = f(x,y) \\ \dfrac{dy}{dt} = g(x,y) \end{cases}$$

Critical Point (x_0,y_0):
$$\begin{cases} f(x_0,y_0) = 0 \\ g(x_0,y_0) = 0 \end{cases}$$

Linearized System at (x_0,y_0): $\vec{x}\,' = \mathbb{A}\,\vec{x}$

where
$$\mathbb{A} = \begin{bmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{bmatrix}\Bigg|_{(x_0,y_0)}$$

The origin for $\vec{x}$ corresponds to (x_0,y_0) of orig. syst.

EX: Competing species: $x = \text{deer}$, $y = \text{elk}$

$$\begin{cases} \dfrac{dx}{dt} = x - x^2 - xy = 0 \\ \dfrac{dy}{dt} = \frac{1}{2}y - \frac{1}{4}y^2 - \frac{3}{4}xy = 0 \end{cases}$$

Step 1: Find Critical Points:

$$\begin{cases} x(1 - x - y) = 0 & (A) \\ y\left(\frac{1}{2} - \frac{1}{4}y - \frac{3}{4}x\right) = 0 & (B) \end{cases}$$

For (A) to hold: $x = 0$ or $(1-x-y) = 0$

Case 1: Take $x = 0$, To make (B) hold

$y = 0$ or $\left(\frac{1}{2} - \frac{1}{4}y - \frac{3}{4}0\right) = 0$

$y = 2$.

Get $\boxed{(0,0), (0,2)}$

Case 2: $1 - x - y = 0$. To get (B),

need $y = 0$ or $\frac{1}{2} - \frac{1}{4}y - \frac{3}{4}x = 0$.

Subcase $y=0$: $1-x-0=0$. Get $x=1$.

$$\boxed{(1,0)}$$

Subcase: $\begin{cases} \frac{1}{2}-\frac{1}{4}y-\frac{3}{4}x=0 \\ 1-x-y=0 \end{cases}$ Get $\boxed{\left(\frac{1}{2},\frac{1}{2}\right)}$

Step 2: Linearize:

$$A=\begin{bmatrix} \frac{\partial}{\partial x}(x-x^2-xy) & \frac{\partial}{\partial y}(x-x^2-xy) \\ \frac{\partial}{\partial x}\left(\frac{1}{2}y-\frac{1}{4}y^2-\frac{3}{4}xy\right) & \frac{\partial}{\partial y}\left(\frac{1}{2}y-\frac{1}{4}y^2-\frac{3}{4}xy\right) \end{bmatrix}\Bigg|_{(x_0,y_0)}$$

$$=\begin{bmatrix} 1-2x-y & -x \\ -3/4\, y & \frac{1}{2}-\frac{1}{2}y-\frac{3}{4}x \end{bmatrix}\Bigg|_{(x_0,y_0)}$$

Step 3: Look at each Crt. Pt.

At $(0,0)$: $A=\begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$ $\lambda=\frac{1}{2},1$ $\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\0\end{pmatrix}$

At $(0,2)$: $A=\begin{bmatrix} -1 & 0 \\ -3/2 & -1/2 \end{bmatrix}$ $\lambda=-\frac{1}{2},-1$ $\begin{pmatrix}0\\1\end{pmatrix},\begin{pmatrix}1\\3\end{pmatrix}$

At $(1,0)$:

$$A = \begin{bmatrix} -1 & -1 \\ 0 & -1/4 \end{bmatrix} \qquad \lambda = -\tfrac{1}{4}, -1$$

$$\begin{pmatrix} 4 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

At $\left(\frac{1}{2}, \frac{1}{2}\right)$:

$$A = \begin{bmatrix} -1/2 & -1/2 \\ -3/8 & -1/8 \end{bmatrix} \qquad \lambda = \frac{-5}{16} \pm \frac{\sqrt{57}}{16}$$

$$\begin{pmatrix} 1 \\ \frac{-3 \pm \sqrt{57}}{8} \end{pmatrix}$$

Saddle pt, Unstable!