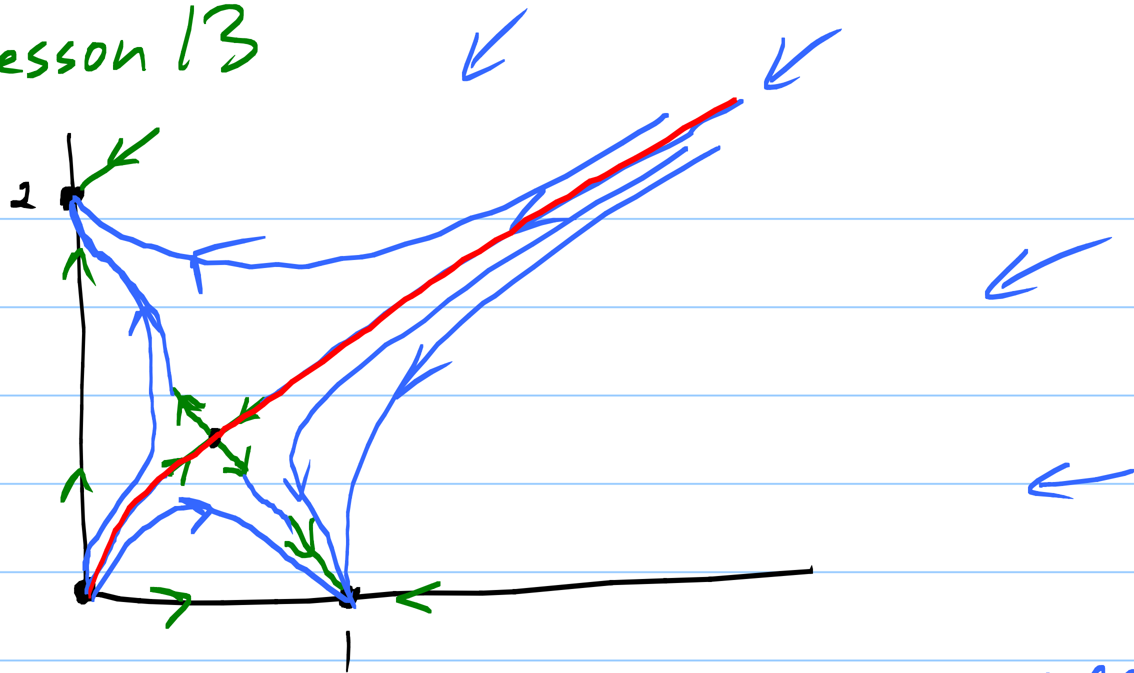
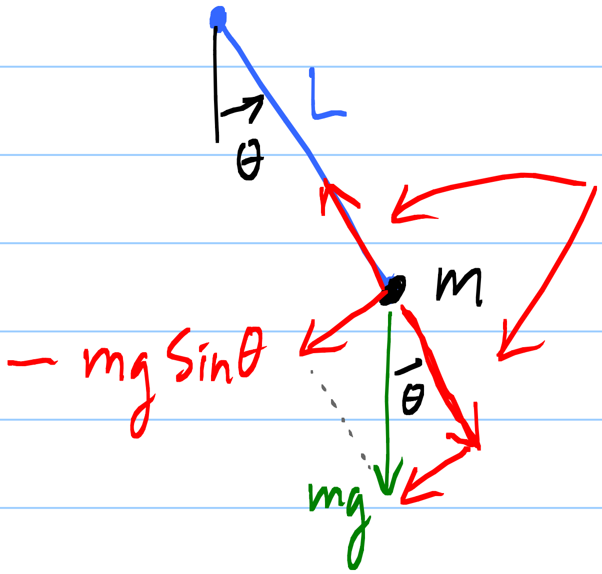
Lesson 13



Damped Pendulum: (with stiff string)



$m\vec{a} = \vec{F}$ ← tangent dir.

$$m(L\ddot{\theta}) = -mg\sin\theta - \underbrace{c\dot{\theta}}_{\text{friction}}$$

$$mL\ddot{\theta} + c\dot{\theta} + mg\underbrace{\sin\theta}_{\approx\theta \text{ for small } \theta} = 0$$

Prob: $\ddot{\theta} = -\sin\theta - \dot{\theta}$ $\quad \begin{cases} x_1 = \theta(t) \\ x_2 = \dot{\theta}(t) \end{cases}$

$$\begin{cases} x_1' = \dot{\theta} = x_2 \\ x_2' = \ddot{\theta} = -\sin\theta - \dot{\theta} = -\sin x_1 - x_2 \end{cases}$$

$$\begin{cases} x_1' = f(x_1, x_2) \\ x_2' = g(x_1, x_2) \end{cases} \qquad \begin{cases} f(x_1, x_2) = x_2 \\ g(x_1, x_2) = -\sin x_1 - x_2 \end{cases}$$

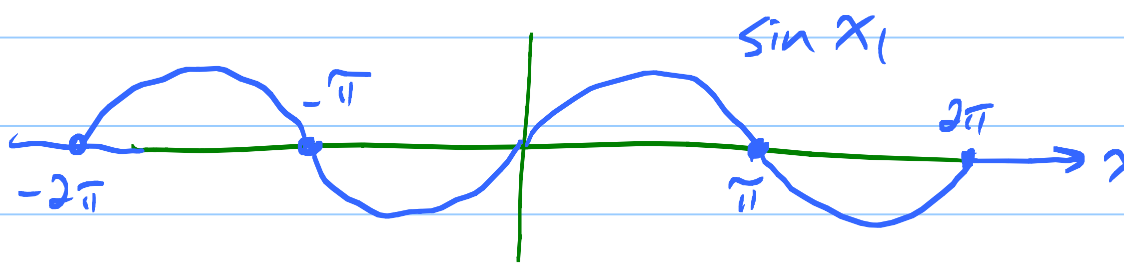
<u>Step 1</u>: Find Crt. Pts. $\begin{cases} f(x_1, x_2) = 0 & (A) \\ g(x_1, x_2) = 0 & (B) \end{cases}$

(A) $x_2 = 0$

(B) $-\sin x_1 - x_2 = 0$

$x_2 = 0$ from (A)

$$\boxed{\begin{matrix} \sin x_1 = 0 \\ x_2 = 0 \end{matrix}}$$



Get Crt. Pts: $\begin{cases} x_1 = n\pi \\ x_2 = 0 \end{cases}$ $n = 0, \pm 1, \pm 2, \ldots$

<u>Step 2</u>: Linearize at Crt. Pts.

Jacobian Matrix $\mathbb{J} = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \\ \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} \end{bmatrix}$

Linearized System: $\vec{x}' = \mathbb{A}\vec{x}$ where

$$\mathbb{A} = J\Big|_{(x_0, y_0)} \leftarrow \text{Crt. Pt.}$$

$f = x_2$

$g = -\sin x_1 - x_2$

$$= \begin{bmatrix} 0 & 1 \\ -\cos x_1 & -1 \end{bmatrix}\Bigg|_{(n\pi, 0)}$$

$$= \begin{bmatrix} 0 & 1 \\ -\cos n\pi & -1 \end{bmatrix}$$

<u>Case</u>: <u>Even</u> multiples of π. $\cos 2n\pi = 1$

$$\mathbb{A} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

e-vals: $(0-\lambda)(-1-\lambda)+1=0$

$\lambda^2 + \lambda + 1 = 0$

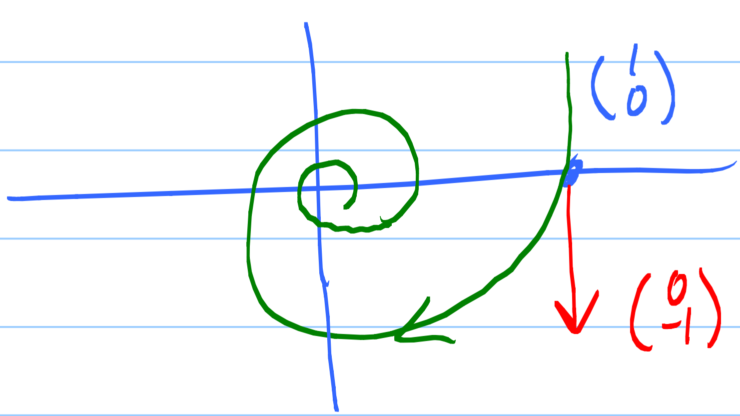
$\lambda = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i$

Re $\lambda < 0$

Spiral in, A. Stable

CW or CCW?

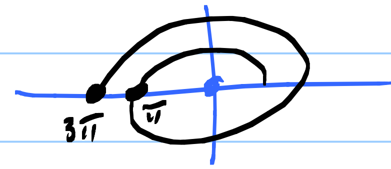
$\vec{x}\,'$ at $\begin{pmatrix}1\\0\end{pmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix}$



Case: Odd mult. of π $\qquad$ $\cos(\text{odd } \pi) = -1$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$$

$-\cos(n\pi)$



$\theta = 0$

e-vals $\quad (0-\lambda)(-1-\lambda) - 1 = 0$

$$\lambda^2 + \lambda - 1 = 0$$

$$\lambda = -\frac{1}{2} \pm \frac{\sqrt{5}}{2}$$

Saddle Pt., Unstable

For $\lambda = -\frac{1}{2} - \frac{\sqrt{5}}{2} < 0$ $\qquad$ $(A - \lambda I)\vec{a} = 0$

$\lambda = -\frac{1}{2} - \frac{\sqrt{5}}{2}$

$$\left[\begin{array}{cc|c} 0 - \left(-\frac{1}{2} - \frac{\sqrt{5}}{2}\right) & 1 & 0 \\ \text{eee} & \text{eee} & 0 \end{array}\right]$$

who cares?

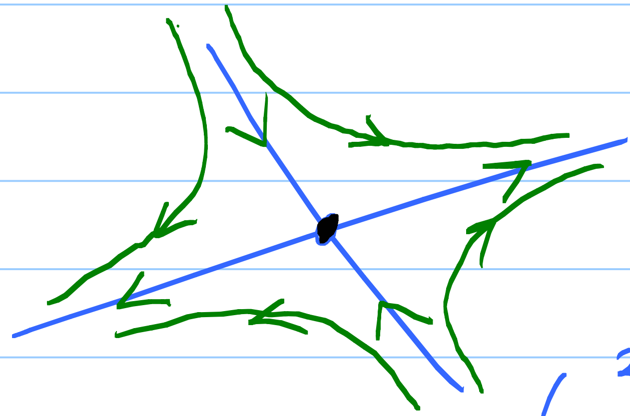
$$\sim \begin{array}{ccc} A & B & \\ \left[\begin{array}{c} \frac{1}{2} + \frac{\sqrt{5}}{2} \\ 0 \end{array}\right. & \begin{array}{c} 1 \\ 0 \end{array} & \left|\begin{array}{c} 0 \\ 0 \end{array}\right] \end{array}$$

faith

e-vect $\quad \begin{pmatrix} 1 \\ -\frac{1}{2} - \frac{\sqrt{5}}{2} \end{pmatrix} \qquad \begin{pmatrix} B \\ -A \end{pmatrix}$

Better e-vect. $\begin{pmatrix} 2 \\ -1-\sqrt{5} \end{pmatrix}$

For $\lambda = -\frac{1}{2} + \frac{\sqrt{5}}{2} > 0$ Get $\begin{pmatrix} 2 \\ -1+\sqrt{5} \end{pmatrix}$



4.6 Non-homogeneous systems:

$\vec{x}\,' = A\vec{x}$ ← linear homog system.

$\vec{x}\,' = A\vec{x} + \vec{g}$ ← non-homog.

Gen^l Sol^n is $\vec{x} = \underbrace{\left(c_1\vec{x}_1 + \cdots + c_n\vec{x}_n\right)}_{\text{Gen}^l \text{ Sol}^n \text{ to homog prob } \vec{x}\,'=A\vec{x}} + \vec{x}_p$

↑ one particular sol^n to $\vec{x}\,' = A\vec{x} + \vec{g}$

EX: $\vec{x}\,' = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}\vec{x} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}e^{2t}$

Step 1: Get homog sol^n: $\vec{x}_c = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix}e^{-2t} + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix}e^{-4t}$

Step 2: Try $\underbrace{\vec{b}\, e^{2t}}_{\vec{x}_p}$. ← safe! because 2 is not an e-val.

Plug in and try to force it to work!

$$\vec{x}_p{}' \overset{\text{want}}{=} \mathbb{A}\,\vec{x}_p + \begin{pmatrix}1\\2\end{pmatrix} e^{2t}$$

$$2\vec{b}e^{2t} = \mathbb{A}\vec{b}e^{2t} + \begin{pmatrix}1\\2\end{pmatrix} e^{2t}$$

Cancel out e^{2t}'s. Get lin. alg. prob.

$$-\begin{pmatrix}1\\2\end{pmatrix} = \mathbb{A}\vec{b} - \underbrace{2\vec{b}}_{2\mathbb{I}\vec{b}}$$

$$\boxed{(\mathbb{A} - 2\mathbb{I})\vec{b} = \begin{pmatrix}-1\\-2\end{pmatrix}}$$

det ≠ 0 because 2 not e-val.

$$\left[\begin{array}{cc|c} -5 & 1 & -1 \\ 1 & -5 & -2 \end{array}\right] \qquad b_1 = \frac{7}{24} \qquad b_2 = \frac{11}{24}$$

Cramer's Rule

$$b_1 = \frac{\det\begin{bmatrix}-1 & 1\\ -2 & -5\end{bmatrix}}{\det\begin{bmatrix}-5 & 1\\ 1 & -5\end{bmatrix}} = \frac{7}{24} \qquad b_2 = \frac{\det\begin{bmatrix}-5 & -1\\ 1 & -2\end{bmatrix}}{\det\begin{bmatrix}-5 & 1\\ 1 & -5\end{bmatrix}} = \frac{11}{24}$$

Now $\vec{x}_p = \vec{b}e^{2t} = \begin{pmatrix} 7/24 \\ 11/24 \end{pmatrix} e^{2t}.$

The Gen^l Sol^n to $\vec{x}' = \mathbb{A}\vec{x} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}$ is

$$\vec{x} = \underbrace{c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-4t}}_{\text{Gen}^l \text{ Sol}^n \text{ to } \mathbb{A}\vec{x} = 0} + \underbrace{\begin{pmatrix} 7/24 \\ 11/24 \end{pmatrix} e^{2t}}_{\text{particular sol}^n \text{ to } \vec{x}' = \mathbb{A}\vec{x} + \binom{1}{2} e^{2t}}$$

$$\begin{cases} x_1 = c_1 e^{-2t} + c_2 e^{-4t} + \frac{7}{24} e^{2t} \\ x_2 = c_1 e^{-2t} - c_2 e^{-4t} + \frac{11}{24} e^{2t} \end{cases}$$

IVP: Find solution satisfying $\vec{x}(0) = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$.

$$\begin{cases} x_1(0) = c_1 + c_2 + \frac{7}{24} \overset{\text{want}}{=} 3 \\ x_2(0) = c_1 - c_2 + \frac{11}{24} = 4 \end{cases}$$

Solve for c_1, c_2.