Lesson 14 4.6

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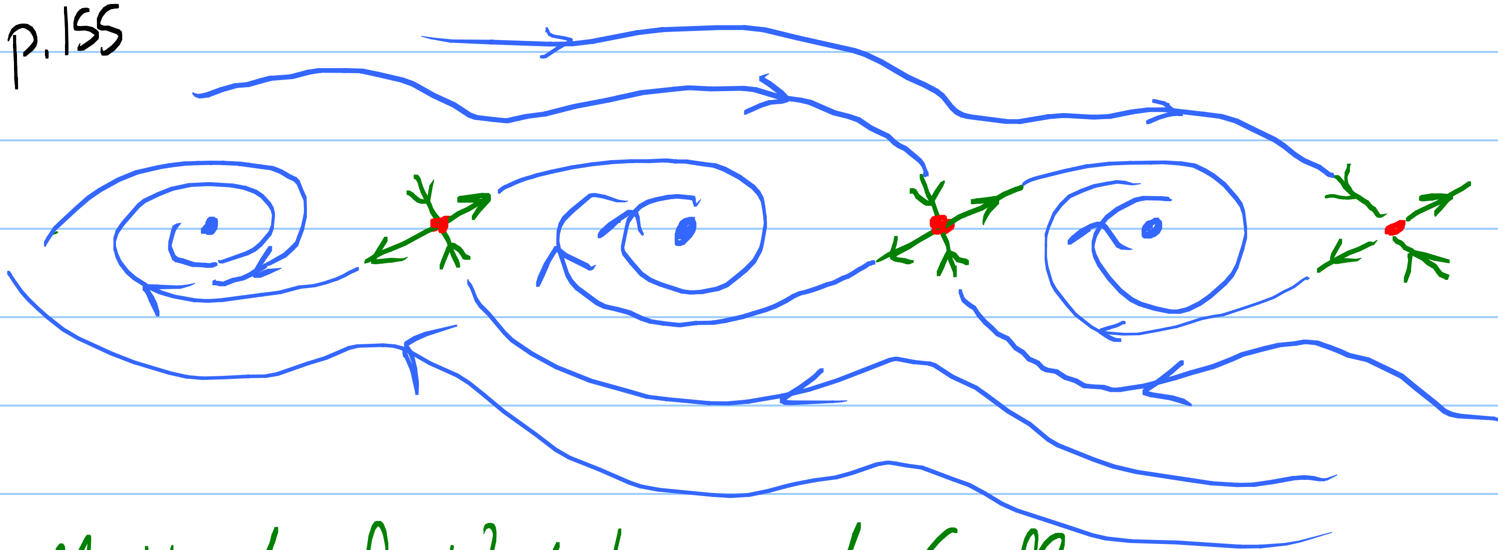

Method of Undetermined Coeff:

$\vec{x}\,' = A\vec{x} + \vec{g}$

constant coeff

$\vec{g}$	Try $\vec{x}_p =$
$\binom{3}{4}e^{at}$	$\vec{b}e^{at}$ ← danger if a = e.val
$\binom{1}{2}t^2$	$\vec{b}_2 t^2 + \vec{b}_1 t + \vec{b}_0$ ← danger if $\lambda = 0$ e-val
$\vec{a}_n t^n + \cdots + \vec{a}_0$	$\vec{b}_n t^n + \cdots + \vec{b}_0$
$\vec{a}\cos\omega t$ $\vec{a}\sin\omega t$ $\vec{a}\cos\omega t + \vec{b}\sin\omega t$	$\vec{b}_1\cos\omega t + \vec{b}_2\sin\omega t$ ← danger if $\pm\omega i$ e-vals

$\vec{a}t^2e^{3t}\sin 4t$ | You guessed it!

Danger: If any little piece of the trial sol^n might solve the homog, then method might bomb.

Variation of Parameters: $\vec{x}\,' = \mathbb{A}\vec{x} + \vec{g}$

coeff can depend on t

Step 1: Solve the homog. sys

$\vec{x}\,' = \mathbb{A}\vec{x}$. Get gen^l sol^n

$$\vec{x}_c = c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n$$

$$= \underbrace{\left[\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\right]}_{\text{Fundamental Matrix}} \underbrace{\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}}_{\text{"parameters"}}$$

$$= \mathbb{X}\vec{c}$$

Step 2: Try $\vec{x}_p = \mathbb{X}\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix} = \mathbb{X}\vec{u}$

turn c's into functions.

Want $\vec{x}_p' = \mathbb{A}\vec{x}_p + \vec{g}$

$$(\mathbb{X}\vec{u})' = \mathbb{A}(\mathbb{X}\vec{u}) + \vec{g}$$

prod. rule $\rightarrow$ $\underbrace{\mathbb{X}'\vec{u}} + \mathbb{X}\vec{u}' = \underbrace{(\mathbb{A}\mathbb{X})\vec{u}} + \vec{g}$

cancel!

$$\mathbb{X}' = [\vec{x}_1', \ldots, x_n'] = [\mathbb{A}\vec{x}_1, \ldots, \mathbb{A}\vec{x}_n]$$

$$= \mathbb{A}[\vec{x}_1, \ldots, \vec{x}_n] = \mathbb{A}\mathbb{X}$$

Left: $\boxed{\mathbb{X}\vec{u}' = \vec{g}}$ $\leftarrow$ big $n \times n$ do row operations

Get u's from here. $\rightarrow$ $\underline{\underline{\vec{u}' = \mathbb{X}^{-1}\vec{g}}}$ $\leftarrow$ 2×2 $\mathbb{X}^{-1}$ easy.

danger!

EX: $\vec{x}' = \begin{pmatrix} -3 & 1 \\ 1 & 3 \end{pmatrix}\vec{x} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t}$

Homog sol^n $\vec{x}_c = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t} + c_2\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-4t}$

$$\mathbb{X} = \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-2t}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-4t}\right]$$

$$\mathbb{X} = \begin{bmatrix} e^{-2t} & e^{-4t} \\ e^{-2t} & -e^{-4t} \end{bmatrix}$$

$$\mathbb{X}^{-1} = \frac{1}{-e^{-6t} - e^{-6t}} \begin{bmatrix} -e^{-4t} & -e^{-4t} \\ -e^{-2t} & e^{-2t} \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

(switch diag minus others)

$$\mathbb{X}^{-1} = \begin{bmatrix} \frac{1}{2}e^{2t} & \frac{1}{2}e^{2t} \\ \frac{1}{2}e^{4t} & -\frac{1}{2}e^{4t} \end{bmatrix}$$

$$\vec{g} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-2t} = \begin{pmatrix} e^{-2t} \\ 2e^{-2t} \end{pmatrix}$$

$$\boxed{\vec{u}\,' = \mathbb{X}^{-1}\vec{g}}$$

$$\vec{u}\,' = \begin{pmatrix} \frac{1}{2}e^{2t}\cdot e^{-2t} + \frac{1}{2}e^{2t}\,2e^{-2t} \\ \frac{1}{2}e^{4t}e^{-2t} - \frac{1}{2}e^{4t}\,2e^{-2t} \end{pmatrix}$$

$$\vec{u}\,' = \begin{pmatrix} 3/2 \\ -\frac{1}{2}e^{2t} \end{pmatrix}$$

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 3/2 \\ -\frac{1}{2}e^{2t} \end{pmatrix}$$

$$u_1 = \int \frac{3}{2}\,dt = \frac{3}{2}t + C$$

dump the $+C$
Just want a part. sol^n.

$$u_2 = \int -\frac{1}{2}e^{2t}\,dt = -\frac{1}{4}e^{2t} + C$$

$$\vec{u} = \begin{pmatrix} \frac{3}{2}t \\ -\frac{1}{4}e^{2t} \end{pmatrix}$$

Done: $\vec{x}_p = \mathbb{X}\vec{u} = \begin{bmatrix} e^{-2t} & e^{-4t} \\ e^{-2t} & -e^{-4t} \end{bmatrix} \begin{pmatrix} \frac{3}{2}t \\ -\frac{1}{4}e^{2t} \end{pmatrix}$

$$= \begin{pmatrix} \frac{3}{2}te^{-2t} - \frac{1}{4}e^{-2t} \\ \frac{3}{2}te^{-2t} + \frac{1}{4}e^{-2t} \end{pmatrix}$$

Aha! Could have tried $\vec{a}te^{-2t} + \vec{b}e^{-2t}$ in Meth. of Undet. Coeff.

Linearized System details: $\begin{cases} \dfrac{dx_1}{dt} = f(x_1, x_2) \\ \dfrac{dx_2}{dt} = g(x_1, x_2) \end{cases}$

Crt. Pts: $\begin{cases} f(x_1, x_2) = 0 \\ g(x_1, x_2) = 0 \end{cases}$

$\vec{x}\,' = \mathbb{A}\vec{x}$ Linearized Sys. at Crt. Pt. (x_0, y_0)

Get e-vals $\lambda_1, \lambda_2 = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

e.vals	Linearized System	Orig. Syst.
$\lambda_2 > \lambda_1 > 0$	Imp. Node, Unst.	Same!
$\lambda_2 < \lambda_1 < 0$	Imp. Node, A. Stable	Same!
$\lambda_1 < 0 < \lambda_2$	Saddle, Unstable	Same!

$\lambda = a \pm bi$, $a<0$	Spiral in, A. Stable, CW or CCW	Same!
$a>0$	Spiral out, Unstable, CW or CCW	Same!
$\lambda = a \pm bi$, $a=0$	Center, Stable	Center <u>or</u> Spiral in <u>or</u> spiral out
$\lambda_1 = \lambda_2$ real	Proper node (2 e-vects) Degenerate node (1-evect.)	Node <u>or</u> Spiral
$\lambda_1 = \lambda_2 < 0$	A. Stable	A. Stable
$\lambda_1 = \lambda_2 > 0$	Unstable	Unstable

Gen^{l} Sol^{n} of $\vec{x}\,' = A\vec{x} + \vec{g}$ is

$$\Big(\underbrace{c_1\vec{x}_1 + \cdots + c_n\vec{x}_n}_{\text{homog sol}^n}\Big) + \vec{x}_p$$

part.

Why: Suppose $\vec{x}$ solves $\vec{x}\,' = \mathbb{A}\vec{x} + \vec{g}$

We have $\vec{x}_p$ $\quad - \quad \vec{x}_p\,' = \mathbb{A}\vec{x}_p + \vec{g}$ cancel

$$(\vec{x} - \vec{x}_p)' = \mathbb{A}(\vec{x} - \vec{x}_p)$$

Aha! $\vec{x} - \vec{x}_p$ solves homog. So

$$\vec{x} - \vec{x}_p = c_1\vec{x}_1 + \cdots + c_n\vec{x}_n$$

for some choice of c's. ✓

Conversely, we need to check that if

$$\vec{x} = c_1\vec{x}_1 + \cdots + c_n\vec{x}_n + \vec{x}_p,$$

then it solves the syst.

$$\left[c_1\vec{x}_1 + \cdots + c_n\vec{x}_n + \vec{x}_p\right]' \overset{?}{=} \mathbb{A}\left[\text{same}\right] + \vec{g}$$

$$\underbrace{c_1\vec{x}_1\,'}_{=} + \cdots + \underbrace{c_n\vec{x}_n\,'}_{=} + \vec{x}_p\,' \overset{?}{=} \underbrace{c_1\mathbb{A}\vec{x}_1}_{=} + \cdots + \underbrace{c_n\mathbb{A}\vec{x}_n}_{=} + \mathbb{A}\vec{x}_p + \vec{g}$$

$$\vec{x}_p\,' = \mathbb{A}\vec{x}_p + \vec{g} \quad ✓$$

So it is a solution. We have the general solution.