Lesson 15 Review (12, 13, 14 due Wed.)

Exam 1 Friday in class.

Salt in T_1 at time 0: 10 lbs

Salt in T_2 at time 0: 0 lbs.

$$\begin{cases} S_1' = (\text{Rate in}) - (\text{Rate out}) \\ \quad = \left(0 \cdot 2\,\text{gal/min} + 2\,\text{gal/min}\,\dfrac{S_2}{100\,g}\right) - (4\,\text{gal/min})\dfrac{S_1}{100\,\text{gal}} \\ S_2' = 4\dfrac{S_1}{100} - 2\,\text{gal/min}\,\dfrac{S_2}{100} - 2 \cdot \dfrac{S_2}{100} \end{cases} \qquad \begin{cases} S_1(0)=10 \\ S_2(0)=0 \end{cases}$$

Old Exam on Home Page: 1. $A = \begin{bmatrix} 8 & 2 & 6 \\ 16 & 6 & 32 \\ 4 & 0 & -7 \end{bmatrix} \overset{?}{\leadsto} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ no!

a) Solve $A\vec{x} = 0$. Write down a basis for the null space.

$$\leadsto \begin{bmatrix} 1 & 0 & -7/4 \\ 0 & 1 & 10 \\ 0 & 0 & 0 \end{bmatrix}$$

reduced REF

basis for row space. 2-dim^e

x_1 bound, x_2 bound, x_3 free

Let $x_3 = t$.

$$\begin{cases} x_1 = \frac{7}{4}x_3 = \frac{7}{4}t \\ x_2 = -10x_3 = -10t \\ x_3 = t \end{cases}$$

$$\vec{x} = \begin{pmatrix} 7/4 \\ -10 \\ 1 \end{pmatrix} t$$

$$\text{Null}(A) = \text{Span}\left\{ \begin{pmatrix} 7 \\ -40 \\ 4 \end{pmatrix} \right\}$$

b) nullity of A? ← Number of free vars. $= 1$

c) rank of A $\quad 2$

d) Does A^{-1} exist? No. $\det(A) = 0$ ← bad.

2. $A = \begin{bmatrix} 4-\lambda & 2 \\ -4 & -2-\lambda \end{bmatrix}$ $\qquad (4-\lambda)(-2-\lambda) + 8 = \lambda^2 - 2\lambda = 0$

$\lambda(\lambda - 2) = 0$

$\lambda = 0, 2$

a) e-vals, e-vects. $\quad (A - \lambda I)\vec{a} = 0$

$\underline{\lambda = 0}$: $\left[\begin{array}{cc|c} 4 & 2 & 0 \\ -4 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\qquad \begin{pmatrix} 1 \\ -2 \end{pmatrix}, \lambda = 0$

$\underline{\lambda = 2}$: $\left[\begin{array}{cc|c} 2 & 2 & 0 \\ -2 & -2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$ $\qquad \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \lambda = 2$

b) Diagonalize A. $\qquad A = P^{-1} D P$

e-vects.

$P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \qquad D = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$

$\lambda = 2 \quad \lambda = 0$ ← match order

(Wish had P orthog. Can if A is symmetric.)

c) $P^{-1} = ?$ $\qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{\det} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

(P orthog: $P^{-1} = P^T$. Not in this prob.)

3. $A = \begin{bmatrix} 0-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ -1 & 0 & 0-\lambda \end{bmatrix}$ $\quad -\lambda^3 + \lambda^2 - \lambda + 1 = 0$

factor out $(\lambda - 1)$

a) Is A orthogonal? Yes. rows $\perp$ and have length one.

b) e-vals? $\leftarrow$ e-vals are ± 1 or conjugate pairs of unit modulus

c) e-vects?

$$\det(A - \lambda I) = (+1)(1-\lambda)\det\begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix}$$

$$= (1-\lambda)(\lambda^2+1) \qquad \lambda = 1, \pm i \quad \checkmark$$

$\underline{\lambda = 1}$: $\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \leftarrow x_1 = x_3 \\ \leftarrow x_3 = 0 \\ \end{matrix}$

$\begin{array}{ccc} x_1 & x_2 & x_3 \\ B & \text{free} & B \end{array}$

Let $x_2 = t$.

$x_1 = x_3 = 0$

$x_3 = 0$

$\vec{x} = \begin{pmatrix} 0 \\ t \\ 0 \end{pmatrix}$. Take $t = 1$, $\boxed{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \lambda = 1}$

$\underline{\lambda = i}$: $\begin{bmatrix} -i & 0 & 1 \\ 0 & 1-i & 0 \\ -1 & 0 & -i \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad \boxed{\begin{pmatrix} -i \\ 0 \\ 1 \end{pmatrix}, \lambda = i}$

$\boxed{\begin{pmatrix} i \\ 0 \\ 1 \end{pmatrix}, \lambda = -i}$

4. $y'' + 25y = 0$

a) First order sytem

$x_1 = y$

$x_2 = y'$

$x_1' = y' = x_2$

$x_2' = y'' = -25y = -25x_1$

$\vec{x}' = \begin{bmatrix} 0 & 1 \\ -25 & 0 \end{bmatrix} \vec{x}$

b) Solve system. Get real sol^n

$$\det\begin{pmatrix} -\lambda & 1 \\ -25 & -\lambda \end{pmatrix} = \lambda^2 + 25 = 0 \qquad \boxed{\lambda = \pm 5i}$$

For $\lambda = 5i$: $(A - \lambda I)\vec{a} = 0$

$$\left[\begin{array}{cc|c} -5i & 1 & 0 \\ -25 & -5i & 0 \end{array}\right]$$

$R2 = -5i\,R1$

$$\left[\begin{array}{cc|c} -5i & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \qquad \begin{pmatrix} 1 \\ 5i \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 5i \end{pmatrix} e^{5it}$$

complex sol^n

$$\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + i\begin{pmatrix} 0 \\ 5 \end{pmatrix}\right](\cos 5t + i \sin 5t)$$

$$= \Bigg[\underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cos 5t - \begin{pmatrix} 0 \\ 5 \end{pmatrix}\sin 5t}_{\vec{x}_1}\Bigg] + i\Bigg[\underbrace{\begin{pmatrix} 0 \\ 5 \end{pmatrix}\cos 5t + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin 5t}_{\vec{x}_2}\Bigg]$$

Real Sol^n $c_1\vec{x}_1 + c_2\vec{x}_2$

$$\boxed{\vec{x}_1 = c_1\begin{pmatrix} \cos 5t \\ -5\sin 5t \end{pmatrix} + c_2\begin{pmatrix} \sin 5t \\ 5\cos 5t \end{pmatrix}}$$

c) Crt. Pt. at $(0,0)$? Center (Stable), λ Pure Imag

d) Phase portrait.

Clockwise.

Test at $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$:

$\vec{x}\,'$ at $\begin{pmatrix} 1 \\ 0 \end{pmatrix} = A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -25 \end{pmatrix}$

$\begin{pmatrix} 0 \\ -25 \end{pmatrix}$

5. $$\frac{dy_1}{dt} = -y_1 + y_2 + y_1 y_2 \qquad = 0$$

$$\frac{dy_2}{dt} = -y_1 - y_2 \qquad = 0$$

a) Find Critical points

$$\boxed{y_2 = -y_1} \quad E2$$

E1: $-y_1 + (-y_1) + y_1(-y_1) = 0$

$-2y_1 - y_1^2 = 0$

$-y_1(y_1 + 2)$

$y_1 = 0, -2$

$$\boxed{\begin{matrix}(0,0)\\(-2,2)\end{matrix}}$$

b) Analize Crt. Pts.

$$\left.\left|\begin{bmatrix} -1+y_2 & 1+y_1 \\ -1 & -1 \end{bmatrix}\right|\right._{(x_0,\,y_0)}$$

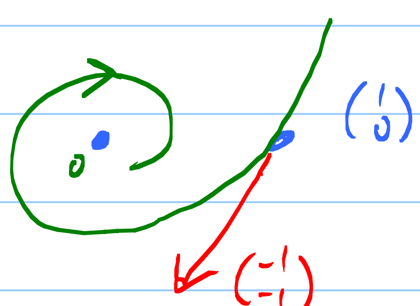
At $(0,0)$:

$$\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}$$

$(-1-\lambda)^2 + 1 = 0 \qquad \lambda = -1 \pm i$

Spiral in, Asympt. Stable

Test field at $\binom{1}{0}$: $\vec{x}\,'$ at $\binom{1}{0} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}\binom{1}{0} = \binom{-1}{-1}$



Clockwise

The non-linear system also has an A. Stable Spiral in (clockwise).

<u>At $(-2,2)$</u>: $\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$

$$\det(A-\lambda I) = (1-\lambda)(-1-\lambda)-1 = \lambda^2-2 = 0$$

$$\lambda = \pm\sqrt{2}$$ Saddle Pt., Unstable

The non-linear system also has a saddle pt (unstable).