Lesson 16 Review.

Exam 1 in class Friday.
Off. Hr. Thurs 1-2

Skype Hour 7-8 EDT

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$$\frac{dx}{dt} = f(x,y)$$

$$\frac{dy}{dt} = g(x,y)$$

(x_0,y_0) Crt. Pt means $\begin{cases} f(x_0,y_0)=0 \\ g(x_0,y_0)=0 \end{cases}$

Easy solⁿ $\begin{cases} x \equiv x_0 \\ y \equiv y_0 \end{cases}$

First row of A

$$f(x,y) = \underset{\overset{\|}{0}}{f(x_0,y_0)} + \frac{\partial f}{\partial x}(x_0,y_0)\underbrace{(x-x_0)}_{\tilde{x}} + \frac{\partial f}{\partial y}(x_0,y_0)\underbrace{(y-y_0)}_{\tilde{y}} + R$$

Vector Space Test:

$S \subset$ Vector Space

Only need to check: 1) $\vec{v}_1, \vec{v}_2$ in S, so is $\vec{v}_1 + \vec{v}_2$

2) $\vec{v}$ in S. Then $c\vec{v}$ is too.

↑ $c=0$ too! Zero vector must be in

$\text{Null}(A) = \{\vec{x} : A\vec{x} = 0\}$.

1) $A\vec{x}_1 = 0$, $A\vec{x}_2 = 0$. $A(\vec{x}_1 + \vec{x}_2) = 0 + 0$ ✓

2) $A\vec{x} = 0$. Then $A(c\vec{x}) = cA\vec{x} = 0$ ✓

1) If $\overset{\swarrow n\times n}{A}$ is diagonalizable, then all its eigenvalues are distinct. T or [✓ F]

Fact: n distinct e-vals $\Rightarrow$ $\underbrace{\text{full set e-vects}}_{\text{equiv to Diagonalizable}}$

EX: I $\lambda = 1, 1, 1$

but I is diagonal!

2) If all the eigenvalues of A are 2, then there is a matrix P so that $P^{-1}AP = 2I$. F

No. Can have a defective matrix with e-vals all 2.

$\begin{pmatrix}2&1\\0&2\end{pmatrix}$ is such a matrix.

3) If $B = C^{-1}AC$, then $B^3 = C^{-1}A^3C$. T

$$B^3 = (C^{-1}A\underbrace{C)(C^{-1}}A\underbrace{C)(C^{-1}}AC) = C^{-1}A^3C \quad ✓$$

4) If A^{-1} exists, then all the eigenvalues of A are non-zero. T

A^{-1} exists means A non-singular $\Rightarrow$ $\det(A) \neq 0$

$\det(A) = \det(A - 0I) \neq 0$

↑ 0 is not an e-val.

5) There is some 3x3 matrix with all entries real but so that no eigenvalue is real.

$\det(A - \lambda I) = $ 3rd degree poly. F

$$A\vec{x} = \begin{pmatrix}a\\b\\c\end{pmatrix} \qquad \left[\begin{array}{c|c}\underbrace{\begin{matrix}1&2&3&4\\2&-1&1&0\\2&9&11&16\end{matrix}}_{A} & \begin{matrix}a\\b\\c\end{matrix}\end{array}\right]$$

↑ hit

if $-4a+b+c=0$

$$\rightsquigarrow \left[\begin{array}{cccc|c}1&2&3&4&a\\0&-5&-5&-8&b-2a\\0&5&5&8&c-2a\end{array}\right]\begin{matrix}\\R2-2R1\\R3-2R1\end{matrix}$$

x_1 bound

x_2 bound

$$\left[\begin{array}{c|c}\underbrace{\begin{matrix}\boxed{1}&2&3&4\\0&\boxed{-5}&-5&-8\\0&0&0&0\end{matrix}}_{E} & \begin{matrix}a\\b-2a\\\boxed{-4a+b+c}\end{matrix}\end{array}\right]\begin{matrix}\\\\R3+R2\end{matrix}$$

↑ free ↑ free

danger!

$\text{Rank}(A) = \#$ non-zero rows in $E = 2$

$\underbrace{\text{Basis for row space of } A}_{\text{2-dim}^e} = \begin{matrix}[1 & 2 & 3 & 4]\\ [0 & 5 & 5 & 8]\end{matrix}$

Basis for the $\underbrace{\text{null space of } A}_{\text{dim } 2}$ $= \begin{pmatrix}-1\\-1\\1\\0\end{pmatrix}, \begin{pmatrix}-4/5\\-8/5\\0\\1\end{pmatrix}$

$x_3=1 \rightarrow$, $x_4=0 \rightarrow$; $\leftarrow x_3=0$, $\leftarrow x_4=1$

#free vars $=$ dim 2

Basis for the column space $= \begin{pmatrix}1\\2\\2\end{pmatrix}, \begin{pmatrix}2\\-1\\9\end{pmatrix}$

$\swarrow$ col 1 $\swarrow$ col 2

x_1 bound x_2 bound

Or stick cols in as rows and row reduce.

Find the general solution to $A\vec{x} = \begin{pmatrix}1\\2\\2\end{pmatrix}$.

↖ first col A

$$\underset{\vec{x}_p}{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}} + \underbrace{c_1\begin{pmatrix}-1\\-1\\1\\0\end{pmatrix} + c_2\begin{pmatrix}-4\\-8\\0\\5\end{pmatrix}}_{\text{homog sol}^n}$$

3. $A = \begin{bmatrix}1&3&4\\0&1&5\\0&0&2\end{bmatrix}$ $\lambda = 1, 1, 2$

no!

$\lambda=1$: $\begin{bmatrix}0&3&4\\0&0&5\\0&0&1\end{bmatrix} \rightsquigarrow \left[\begin{array}{ccc|c}0&\boxed{3}&4&0\\0&0&\boxed{1}&0\\0&0&0&0\end{array}\right]$

$\swarrow x_1$ free

Only get one e-vect for $\lambda=1$

yes! $\mathbb{B} = \begin{bmatrix} 1-\lambda & 4 & 5 \\ 0 & 2-\lambda & 6 \\ 0 & 0 & 3-\lambda \end{bmatrix}$ $\det = (1-\lambda)(2-\lambda)(3-\lambda)$

$\lambda = 1, 2, 3$ distinct.

Full set e-vects.

yes! $\mathbb{C} = \begin{bmatrix} \textcircled{1-\lambda} & 0 & 0 \\ 0 & -1-\lambda & 2 \\ 0 & -3 & 4-\lambda \end{bmatrix}$ $\det(\mathbb{C} - \lambda \mathbb{I}) = (1-\lambda)\left[(-1-\lambda)(4-\lambda) + 6\right]$

$$= (1-\lambda)\left[\underbrace{\lambda^2 - 3\lambda + 2}_{(\lambda-1)(\lambda-2)}\right]$$

$$\boxed{\lambda = 1, 1, 2}$$

For $\lambda = 1$.

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & -2 & 2 \\ 0 & -3 & 3 \end{bmatrix} \sim \begin{bmatrix} 0 & \textcircled{1} & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad E1$$

x_1 — x_3 — free

Let $x_1 = t_1$

$x_3 = t_2$

E1: $x_2 = x_3 = t_2$

List: $\begin{cases} x_1 = t_1 \\ x_2 = t_2 \\ x_3 = t_2 \end{cases}$ $\vec{x} = t_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + t_2 \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$

two e-vects.

5. $\mathbb{A} = \begin{bmatrix} 0 & 5 \\ -1 & -2 \end{bmatrix}$ e.val $\lambda = \textcircled{-1} + 2i$ Spiral In.

$\vec{x}' = \mathbb{A}\vec{x}$ e.vect $\begin{pmatrix} 5 \\ -1+2i \end{pmatrix}$ Asymp. Stable

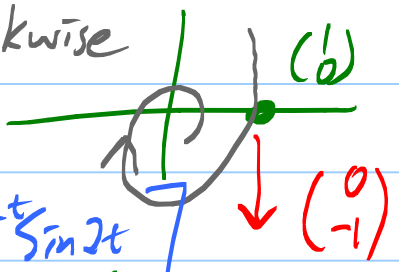
$\vec{x}'$ at $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$= \begin{bmatrix} 0 & 5 \\ -1 & -2 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$$\vec{x} = \begin{pmatrix} 5 \\ -1+2i \end{pmatrix} e^{(-1+2i)t} \leftarrow \text{complex sol}^n$$

$$= \left[\begin{pmatrix} 5 \\ -1 \end{pmatrix} + i \begin{pmatrix} 0 \\ 2 \end{pmatrix}\right]\left(e^{-t}\cos 2t + i e^{-t}\sin 2t\right)$$

clockwise



$$= \underbrace{\left[\begin{pmatrix} 5 \\ -1 \end{pmatrix} e^{-t}\cos 2t - \begin{pmatrix} 0 \\ 2 \end{pmatrix} e^{-t}\sin 2t\right]}_{\vec{x}_1} + i \underbrace{\left[\begin{pmatrix} 0 \\ 2 \end{pmatrix} e^{-t}\cos 2t + \begin{pmatrix} 5 \\ -1 \end{pmatrix} e^{-t}\sin 2t\right]}_{\vec{x}_2}$$

$\uparrow i^2 = -1$

two real sol^{n}s!

$$\vec{x} = c_1 e^{-t}\begin{pmatrix} 5\cos 2t \\ -\cos 2t - 2\sin 2t \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 5\sin 2t \\ 2\cos 2t - \sin 2t \end{pmatrix}$$

6. $y'' + y - y^2/2 = 0$ $x_1 = y$

$x_2 = y'$

$$\begin{cases} x_1' = y' = x_2 \\ x_2' = y'' = -y + \frac{y^2}{2} = -x_1 + \frac{x_1^2}{2} \end{cases}$$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} x_2 \\ -x_1 + \frac{x_1^2}{2} \end{pmatrix}$$ non-linear system.

Step 1: Crt. Pts. $\begin{cases} x_2 = 0 \quad ✓ \\ -x_1 + \frac{x_1^2}{2} = 0 \end{cases}$

$x_1\left(-1 + \frac{x_1}{2}\right) = 0$ $x_1 = 0,\ 2$

Crt. Pts. $(0,0)$, $(2,0)$

At $(0,0)$: $J\Big|_{(0,0)} = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix}_{(0,0)}$

$= \begin{bmatrix} 0 & 1 \\ -1+x_1 & 0 \end{bmatrix}\Big|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$\det(A - \lambda I) = \lambda^2 + 1 = 0$. $\lambda = \pm i$ Center, stable.

Non-linear system could have a center or a spiral in or a spiral out.

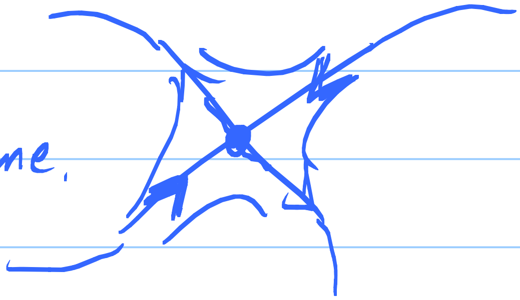
At (2,0) : $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

$\lambda^2 - 1 = 0$

$\lambda = \pm 1$

Saddle Pt. Unstable.

Non-linear system has same.



If $\mathbb{A} \sim \mathbb{B}$ and $\mathbb{B} \sim \mathbb{C}$, show that $\mathbb{A} \sim \mathbb{C}$.

$\mathbb{A} = \mathbb{P}^{-1}\mathbb{B}\mathbb{P}$ $\qquad$ $\mathbb{B} = \underbrace{\mathbb{Q}^{-1}\mathbb{C}\mathbb{Q}}$

$\mathbb{A} = \mathbb{P}^{-1}(\mathbb{Q}^{-1}\mathbb{C}\mathbb{Q})\mathbb{P}$

Let $\mathbb{R} = \mathbb{Q}\mathbb{P}$.

Then $\mathbb{R}^{-1} = \mathbb{P}^{-1}\mathbb{Q}^{-1}$

Aha! $\mathbb{A} = \underbrace{(\mathbb{P}^{-1}\mathbb{Q}^{-1})}_{\mathbb{R}^{-1}} \mathbb{C} \underbrace{(\mathbb{Q}\mathbb{P})}_{\mathbb{R}}$

$\mathbb{A} \sim \mathbb{C}$.