Lesson 18 Laplace Transf. 18, 19, 20 due Fri., Oct. 11

$$\mathcal{L}[\underbrace{f(t)}_{t\geq 0}] = \int_0^\infty e^{-st} f(t)\,dt = F(s)$$

Huge fact 1: $\boxed{\mathcal{L}[f'(t)] = sF(s) - f(0)}$

$$\mathcal{L}[f''(t)] = s\mathcal{L}[f'(t)] - f'(0)$$

$$\boxed{\mathcal{L}[f''(t)] = s^2F(s) - sf(0) - f'(0)}$$

Huge fact 2: If $\mathcal{L}[f_1] = \mathcal{L}[f_2]$ for $s > M$, then $f_1 = f_2$! Can invert $\mathcal{L}$!

something

Master plan: ODE $y'' + y = f(t)$, $\begin{cases} y(0) = y_0 \\ y'(0) = y_0' \end{cases}$

Hit it with $\mathcal{L}$: $(s^2Y - sy_0 - y_0') + Y = F$

Use algebra to get Y. Undo $\mathcal{L}$ to get y.

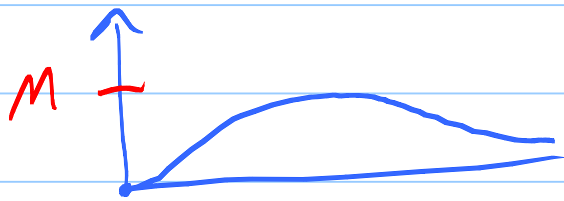
$$\mathcal{L}[\text{ODE}] = \text{Algebra} \xrightarrow{\mathcal{L}^{-1}} \text{solution}.$$

Requirements for $\mathcal{L}[f]$ to make sense:

1) f needs to be "piecewise continuous" on $[0, \infty)$.

2) f needs to be of "exponential type," meaning there are positive constants M and K such that $|f(t)| \le Me^{Kt}$ for $t \ge 0$.

<u>EX</u>: $t^2 e^{3t} \cos 2t$ $\qquad \dfrac{t^2}{e^t} \longrightarrow 0$ as $t \to \infty$



$t^2 \le Me^t$

$$\left| t^2 e^{3t} \cos 2t \right| \le \underbrace{Me^t e^{3t}}_{Me^{4t}} \checkmark$$

<u>EX</u>: $e^{(e^t)}$ <u>not</u> of exp type!

<u>EX</u>: $\mathcal{L}[e^{at}] = \int_0^\infty e^{-st} [e^{at}]\, dt$

$$= \int_0^\infty e^{(a-s)t}\, dt = \lim_{B \to \infty} \int_0^B e^{(a-s)t}\, dt$$

$$= \lim_{B \to \infty} \left[\frac{1}{(a-s)} e^{(a-s)t} \right]_0^B$$

$$= \lim_{B\to\infty}\left[\frac{1}{(a-s)}e^{(a-s)B} - \frac{1}{a-s}\right] = 0 - \frac{1}{a-s}$$

goes away exactly when $a-s<0$

$\boxed{s>a}$

Table entries: $\mathcal{L}[e^{at}] = \frac{1}{s-a}$ for $s>a$.

$$\mathcal{L}[\sin(bt)] = \frac{b}{s^2+b^2} \qquad (s>0)$$

$$\mathcal{L}[\cos(bt)] = \frac{s}{s^2+b^2} \qquad (s>0)$$

$$\mathcal{L}[1] = \frac{1}{s}$$

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}[\sinh(bt)] = \mathcal{L}\left[\tfrac{1}{2}e^{bt} - \tfrac{1}{2}e^{-bt}\right]$$

$$= \frac{1}{2}\cdot\frac{1}{s-b} - \frac{1}{2}\cdot\frac{1}{(s-(-b))} = \frac{b}{s^2-b^2}$$

The s-shifting rule: If $\mathcal{L}[f(t)] = F(s)$,

then $\mathcal{L}[e^{at}f(t)] = F(s-a)$.

Why: $F(s) = \int_0^\infty e^{-st} f(t)\,dt$

$$F(s-a) = \int_0^\infty e^{-(s-a)t} f(t)\,dt$$

$$= \int_0^\infty e^{-st} \left[e^{at} f(t) \right] dt$$

$$= \mathcal{L}\left[e^{at} f(t) \right] \checkmark$$

Now

$$\boxed{\mathcal{L}\left[e^{at} \sin bt\right] = \frac{b}{(s-a)^2 + b^2}}$$

$$\boxed{\mathcal{L}\left[e^{at} \cos bt\right] = \frac{(s-a)}{(s-a)^2 + b^2}}$$

Tricks: Partial Fractions, completing square.

Completing the square: $As^2 + Bs + C$

Step 1: Factor out A: $A\left(\underbrace{s^2 + \frac{B}{A}s + \frac{C}{A}}_{s^2 + bs + c}\right)$

Step 2: $s^2 + bs + c = \left(s + \frac{b}{2}\right)^2 + \left(c - \frac{b^2}{4}\right)$

take $\frac{1}{2}b$

Prob: Find $\mathcal{L}^{-1}\left[\frac{2s}{s^2 - 4s + 13}\right]$ complex roots, complete the square

$$s^2-4s+13 = (s-2)^2 + 9$$ ← Think $9=b^2$ from table.

$$= (s-2)^2 + 3^2$$

$$\frac{2s}{(s-2)^2+3^2} = \frac{2\left[(s-2)+2\right]}{(s-2)^2+3^2}$$

$$= 2\cdot\frac{(s-2)}{(s-2)^2+3^2} + \frac{4}{3}\,\frac{3}{(s-2)^2+3^2}$$

$$= \mathcal{L}\Big[\underbrace{2e^{2t}\cos 3t + \tfrac{4}{3}e^{2t}\sin 3t}_{\text{answer}}\Big]$$

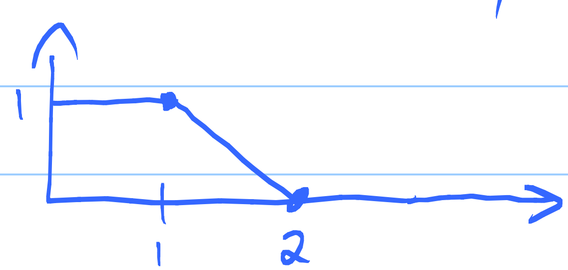
EX: $$\frac{s+4}{s^2+4s+3} = \frac{s+4}{(s+1)(s+3)} = \frac{A}{s+1} + \frac{B}{s+3}$$

real roots

partial fractions!

$$\mathcal{L}\Big[\underbrace{Ae^{-t} + Be^{-3t}}_{\text{answer}}\Big]$$

Virtue of $\mathcal{L}$: Hardly notices piecewise definedness!



$$f(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 2-t & 1 \le t \le 2 \\ 0 & t \ge 2 \end{cases}$$

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st}\, f(t)\, dt$$

$$= \int_0^1 e^{-st}\cdot 1\, dt + \int_1^2 e^{-st}(2-t)\,dt + \int_2^\infty e^{-st}\cdot 0\, dt$$

$$= \frac{1}{s} + \frac{e^{-2s}}{s^2} - \frac{e^{-s}}{s^2}$$

Integration by parts:

EX: $I = \int \underbrace{t}_{u}\ \underbrace{e^{-st}\, dt}_{dv}$

$u = t \qquad du = dt$

$dv = e^{-st}\, dt$

$v = \int e^{-st}\, dt = -\frac{1}{s} e^{-st}$

no $+C$

$$\boxed{\int u\, dv = uv - \int v\, du}$$

$$I = (t)\left(-\frac{1}{s} e^{-st}\right) - \int \left(-\frac{1}{s} e^{-st}\right) dt$$

$$= -\frac{t}{s} e^{-st} - \frac{1}{s^2} e^{-st} + C$$

$$\int_a^b u\, dv = uv\Big|_a^b - \int_a^b v\, du$$