Lesson 19 6.2 18, 19, 20 due Fri., Oct. 11

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty e^{-st} f(t)\,dt$$

$$\mathcal{L}[f'(t)] = sF(s) - f(0)$$

Why: $\int_0^B \underbrace{e^{-st}}_{u} \underbrace{f'(t)\,dt}_{dv}$

$u = e^{-st}$ $du = -se^{-st}$

$dv = f'(t)dt$ $v = \int f'(t)dt = f(t)$

$$= uv\Big|_0^B - \int_0^B v\,du = \left(e^{-st} f(t)\right)\Big|_0^B - \int_0^B \underbrace{f(t)}_{v} \underbrace{(-se^{-st})dt}_{du}$$

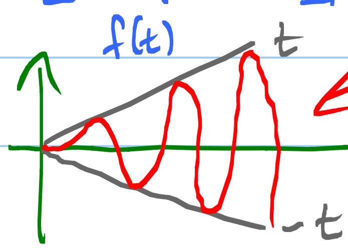
$$= e^{-sB} \underbrace{f(B)}_{|f(B)| \le Me^{kB}} - f(0) + s \underbrace{\int_0^B e^{-st} f(t)\,dt}_{\to \mathcal{L}[f] = F}$$

First term $\to 0$ if $s > k$. As $B \to \infty$, get

$$= 0 - f(0) + sF(s)$$ ✓

$$\mathcal{L}[f^{(n)}(t)] = s^n F(s) - f(0)s^{n-1} - f'(0)s^{n-2} - \cdots - f^{(n-2)}(0)s - f^{(n-1)}(0)$$

EX: $\mathcal{L}[\underbrace{t\sin t}_{f(t)}] = F(s)$



resonance!

$t\sin t$

$f(0) = 0$

$f'(0) = 0$

Trick: $f'(t) = t\cos t + \sin t$ $\leftarrow f'(0)=0$

$$f''(t) = \underbrace{-t\sin t}_{\text{Hmmm.}} + 2\cos t$$

$$\mathcal{L}[f''(t)] = s^2F(s) - s\underbrace{f(0)}_{0} - \underbrace{f'(0)}_{0} = s^2F(s)$$

$$= \mathcal{L}[-t\sin t + 2\cos t] = -F(s) + 2\frac{s}{s^2+1}$$

Equate, solve for $F(s)$: $s^2F(s) = -F(s) + \frac{2s}{s^2+1}$

$$(s^2+1)F(s) = \frac{2s}{s^2+1} \qquad F(s) = \frac{2s}{(s^2+1)^2}$$

Fact: $\mathcal{L}\left[\underbrace{\int_0^t f(u)\,du}_{g(t)}\right] = \frac{1}{s}\underbrace{F(s)}_{\mathcal{L}[f(t)]}$

Fund. Thm. Calculus: $g'(t) = f(t)$

$$F(s) = \mathcal{L}[f(t)] = \mathcal{L}[g'(t)] = s\mathcal{L}[g(t)] - \underbrace{g(0)}_{0}$$

$$g(0) = \int_0^0 f(u)\,du = 0$$

$$\mathcal{L}[g(t)] = \frac{1}{s}F(s) \checkmark$$

EX: $\mathcal{L}^{-1}\left[\frac{1}{s(s^2+1)}\right] = \,? = 1-\cos t$

$$\frac{1}{s}\cdot\underbrace{\frac{1}{s^2+1}}_{F(s)} = \mathcal{L}\left[\int_0^t \sin u\, du\right]$$

Then $f(t) = \sin t$

$$= \mathcal{L}\left[(-\cos u)\Big|_0^t\right]$$

$$= \mathcal{L}\left[(-\cos t) - (-\cos 0)\right]$$

$$= \mathcal{L}\left[\underbrace{1-\cos t}_{\text{ans.}}\right]$$

EX: Another way: $\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$

Step 1: Multiply by denom: $s(s^2+1)$

$$1 = A(s^2+1) + s(Bs+C)$$

Step 2: Collect coeff of powers of s:

$$0\cdot s^2 + 0\cdot s + 1 = (A+B)s^2 + (C)s + (A)$$

Step 3: Equate coeff:

$A+B=0$ $\quad B=-A=-1$

$C=0$

$A=1$

Done: $\dfrac{1}{s(s^2+1)} = \dfrac{1}{s} + \dfrac{(-1)s+0}{s^2+1} = \dfrac{1}{s} - \dfrac{s}{s^2+1}$

$= \mathcal{L}\left[1 - \cos t\right]$ ✓

Partial Fractions: $\dfrac{s^9+5s^3+s+1}{s^2(s-1)^3(s-3)\underbrace{(s^2+2s+2)}_{\text{complex roots}}{}^2}$ deg Num < deg Denom

$= \underbrace{\dfrac{A}{s^2} + \dfrac{B}{s}}_{\text{for } s^2} + \underbrace{\dfrac{C}{(s-1)^3} + \dfrac{D}{(s-1)^2} + \dfrac{E}{(s-1)}}_{\text{for } (s-1)^3} + \underbrace{\dfrac{F}{(s-3)}}_{\text{for } (s-3)} +$

$s = (s-0)$

$+ \underbrace{\dfrac{Gs+H}{(s^2+2s+2)^2} + \dfrac{Is+J}{(s^2+2s+2)}}_{\text{for } (s^2+2s+2)^2}$ MAPLE ?parfrac

EX: $y'' + 4y = e^{2t}$ $\begin{cases} y(0)=1 \\ y'(0)=0 \end{cases}$

Think: Circuit e^{2t} = Input, y = Output

Spring-mass e^{2t} = Forcing, y = Response

Hit ODE with $\mathcal{L}$:

$$\mathcal{L}[y''] + 4\mathcal{L}[y] = \mathcal{L}[e^{2t}]$$

$$\left(s^2 Y - s\underbrace{y(0)}_{1} - \underbrace{y'(0)}_{0}\right) + 4Y = \frac{1}{s-2}$$

$$(s^2+4)Y = s + \frac{1}{s-2}$$

$$Y = \underbrace{\frac{1}{s^2+4}}_{\text{Transfer function}}\left[s + \frac{1}{s-2}\right]$$

$$Y = \frac{s}{s^2+4} + \frac{1}{(s-2)(s^2+4)}$$

$$\uparrow \qquad\qquad \uparrow$$

$$\mathcal{L}[\cos 2t] \qquad\qquad ?$$

$$\frac{1}{(s-2)(s^2+4)} = \frac{A}{s-2} + \frac{Bs+C}{s^2+4}$$

1. $1 = A(s^2+4) + (s-2)(Bs+C)$

2. $1 = \underbrace{(A+B)}_{0}s^2 + \underbrace{(C-2B)}_{0}s + \underbrace{(4A-2C)}_{1}$

3. $\begin{cases} A+B=0 \\ C-2B=0 \\ 4A-2C=1 \end{cases}$ $\qquad \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 0 \\ 4 & 0 & -2 & 1 \end{array}\right] \sim$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/8 \\ 0 & 1 & 0 & -1/8 \\ 0 & 0 & 1 & -1/4 \end{array}\right]$$

$A = 1/8$
$B = -1/8$
$C = -1/4$

$$Y = \frac{s}{s^2+4} + \frac{1/8}{s-2} + \frac{(-1/8)s - 1/4}{s^2+4}$$

$$Y = \frac{7}{8}\,\frac{s}{s^2+4} - \frac{1}{4}\cdot\frac{1}{2}\,\frac{2}{s^2+4} + \frac{1/8}{s-2}$$

$$y = \underbrace{\frac{7}{8}\cos 2t - \frac{1}{8}\sin 2t}_{\text{a homog sol}^n} + \underbrace{\frac{1}{8}e^{2t}}_{\text{part. sol}^n}$$

Hwk hints: p. 216 : 17.

$f(t) = te^{at}$

$f'(t) = ate^{at} + e^{at}$

$\mathcal{L}[f'(t)] =$ One way / another way — equate

p. 216 : 18

$f(t) = \sin^2 \omega t$

$f'(t) = 2\sin\omega t\cos\omega t = \sin 2\omega t$

$\mathcal{L}[f'(t)] =$ One / Two — equate

No no : $\mathcal{L}[f(t)g(t)] = F(s)G(s)$ no!

To understand $\mathcal{L}^{-1}$ you <u>must</u> let s become a <u>complex</u> variable.