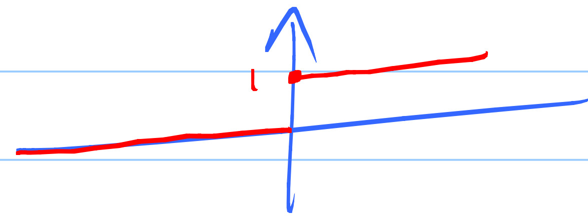
Lesson 20 6.3 Heaviside function

(No lecture on Monday - Oct. Break)

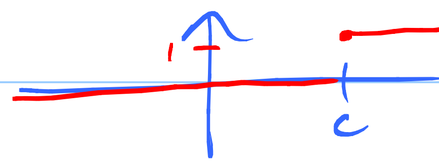
18, 19, 20 due Fri., Oct. 11.

Step Function:

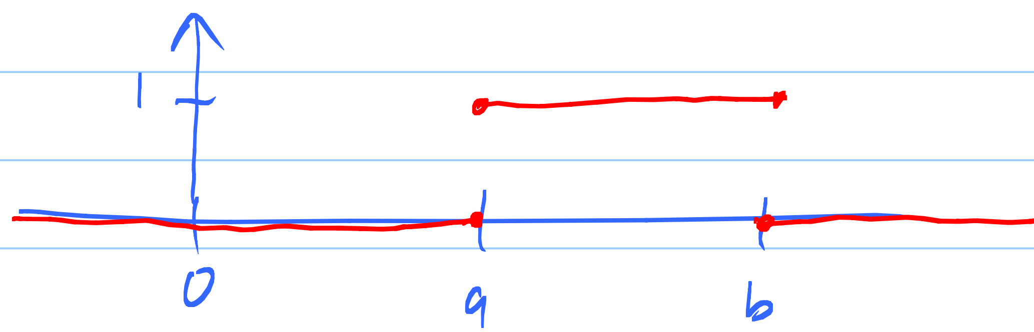


$u(t)$

Turn on at time c:



$u(t-c)$

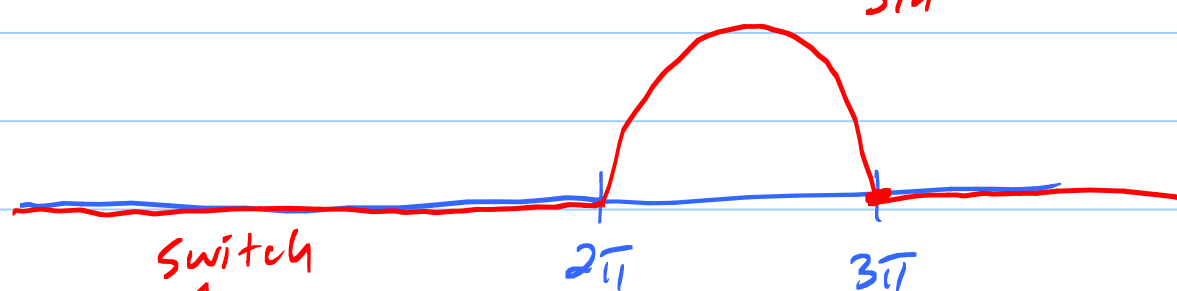


$u(t-a) - u(t-b)$

on at a

off at b

EX:



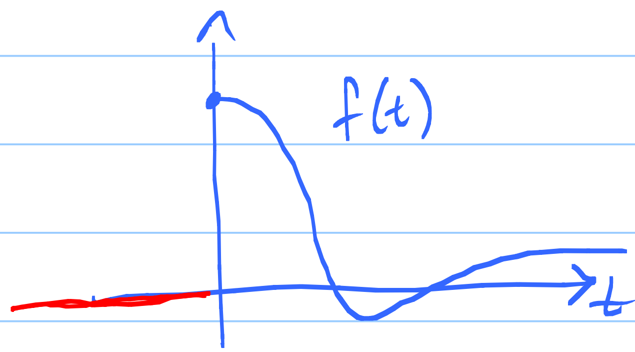
$$\underbrace{\left[u(t-2\pi) - u(t-3\pi)\right]}^{\text{switch}} \sin t$$

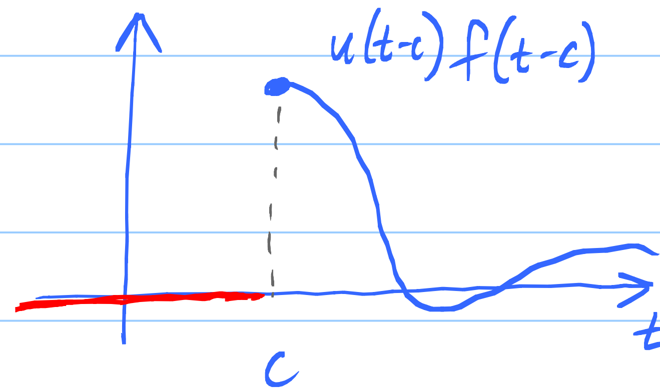
Two Important Facts:

1) s-shifting formula: $\mathcal{L}[e^{at}f(t)] = F(\underbrace{s-a}_{\text{shift}})$

1) t-shifting formula: $\mathcal{L}[\underbrace{u(t-c)f(t-c)}_{\text{shift}}] = e^{-cs}F(s)$

Take $f \equiv 1$. See $\boxed{\mathcal{L}[u(t-c)] = \dfrac{e^{-cs}}{s}}$.





Why: $\mathcal{L}[u(t-c)f(t-c)] = \int_0^\infty e^{-st}\underbrace{u(t-c)}_{\begin{cases}0, & t<c\\ 1, & t>c\end{cases}} f(t-c)$

$$= \int_c^\infty e^{-st}\cdot \underline{1}\, f(\underbrace{t-c}_{\tilde{\tau}})\,dt$$

$$= \int_{\tilde{\tau}=0}^\infty e^{-s(\overbrace{\tilde{\tau}+c}^{=t})} f(\tilde{\tau})\,d\tilde{\tau}$$

$$= \int_0^\infty e^{-s\tilde{\tau}}\underbrace{e^{-cs}}_{\text{pull out}} f(\tilde{\tau})\,d\tilde{\tau} = e^{-cs}F(s) \checkmark$$

$\tilde{\tau} = t-c$, $\underline{t = \tilde{\tau}+c}$

$d\tilde{\tau} = dt$

When $t=c$, $\tilde{\tau} = \underline{\underline{0}}$

As $t \to \infty$, so does $\tilde{\tau}$.

Trick: $t = [(t-c) + c]$

EX: $\mathcal{L}[u(t-3)\,t] = ?$

$$u(t-3)\,t = u(t-3)\,[\overbrace{(t-3)+3}^{t}]$$

$$= u(t-3)\underbrace{(t-3)}_{f(t-3)} + 3u(t-3)$$

So $f(t) = t$

$$\mathcal{L} = e^{-3s}\frac{1}{s^2} + 3e^{-3s}\frac{1}{s}$$

$\uparrow \mathcal{L}[t]$ $\qquad$ $\uparrow \mathcal{L}[1]$

EX: $u(t-3)t^2 = u(t-3)\,[(t-3)+3]^2$

$$= u(t-3)(t-3)^2 + 6u(t-3)(t-3) + 9u(t-3)$$

$$\mathcal{L} = e^{-3s}\frac{2}{s^3} + 6e^{-3s}\frac{1}{s^2} + 9e^{-3s}\frac{1}{s}$$

$\uparrow \mathcal{L}[t^2]$

EX: $u(t-\pi)\sin t = u(t-\pi)\sin[(t-\pi)+\pi]$

$$\sin(\alpha+\beta) = \cos\alpha\sin\beta + \sin\alpha\cos\beta$$

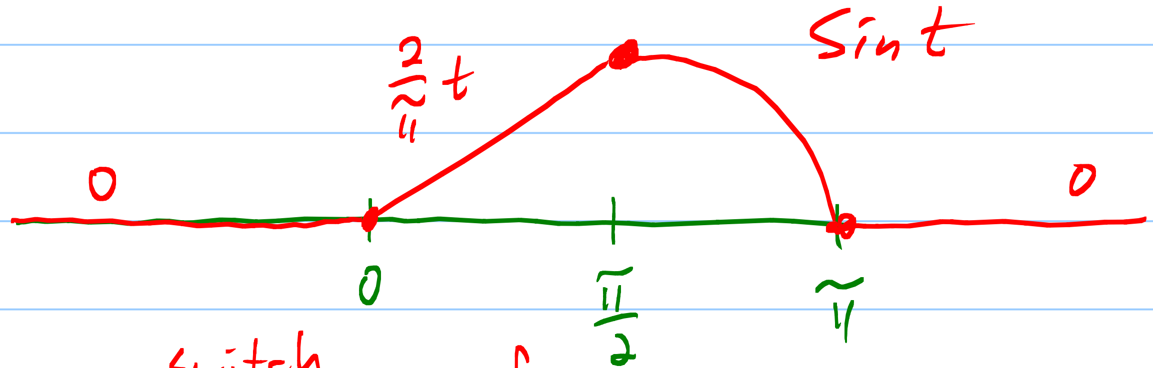
$$\sin((t-\pi)+\pi) = \cos(t-\pi)\underbrace{\cancel{\sin\pi}}_{0} + \sin(t-\pi)\underbrace{\cos\pi}_{=-1}$$

$$= -\sin(t-\pi)$$

$$u(t-\pi)\sin t = -u(t-\pi)\sin(t-\pi)$$

$$\mathcal{L} = -e^{-\pi s}\frac{1}{s^2+1}$$

EX:

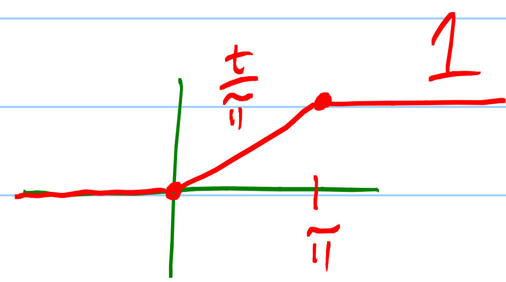


$$\underbrace{\left[u(t)-u\left(t-\tfrac{\pi}{2}\right)\right]}_{\text{switch}}\underbrace{\left(\tfrac{2}{\pi}t\right)}_{\text{fcn}} + \left[u\left(t-\tfrac{\pi}{2}\right)-u(t-\pi)\right]\sin t$$

on at 0   off at $\frac{\pi}{2}$

EX:

$$y''+y = g(t)$$



$$\begin{cases} y(0)=0 \\ y'(0)=0 \end{cases}$$

$$g(t) = [u(t)-u(t-\pi)]\frac{t}{\pi} + u(t-\pi)\cdot 1$$

$$= \frac{1}{\pi}u(t-0)(t-0) - \frac{1}{\pi}u(t-\pi)\big[\overbrace{(t-\pi)+\pi}^{t}\big] + u(t-\pi)$$

$$g(t) = \frac{1}{\pi}u(t)t - \frac{1}{\pi}u(t-\pi)(t-\pi) \leftarrow f(t-\pi) = t-\pi$$

$$f(t) = t$$

$$G(s) = \frac{1}{\pi}\frac{1}{s^2} - \frac{1}{\pi} e^{-\pi s} \underbrace{\frac{1}{s^2}}_{\mathcal{L}[t]}$$

$$\mathcal{L}[y'' + y] = \mathcal{L}[g]$$

$$\left(s^2 Y - s\underbrace{y(0)}_{=0} - \underbrace{y'(0)}_{=0}\right) + Y = \frac{1}{\pi}\frac{1}{s^2}\left(1 - e^{-\pi s}\right)$$

$$Y = \underbrace{\frac{1}{\pi}\frac{1}{s^2(s^2+1)}}_{H(s)}\left(1 - e^{-\pi s}\right)$$

$$Y = H(s) + e^{-\pi s} H(s)$$

$$\boxed{y = h(t) + u(t-\pi)\,h(t-\pi)}$$

$$H(s) = \frac{1}{\pi}\frac{1}{s^2(s^2+1)} \overset{\text{Partial Fractions}}{=} \frac{A}{s^2} + \frac{B}{s} + \frac{Cs+D}{s^2+1} \overset{\text{do it}}{=} \frac{(1/\pi)}{s^2} - \frac{(1/\pi)}{s^2+1}$$

So $\quad h(t) = \frac{1}{\pi} t - \frac{1}{\pi}\sin t$

Now $y(t) = h(t) - u(t-\pi)\,h(t-\pi)$

$$= \frac{1}{\pi}(t - \sin t) - u(t-\pi)\frac{1}{\pi}\left((t-\pi) - \underbrace{\sin(t-\pi)}_{= -\sin t}\right)$$

$$= \begin{cases} \frac{1}{\pi}(t - \sin t) & 0 \le t \le \pi \\ 1 - \frac{2}{\pi}\sin t & t > \pi \end{cases}$$