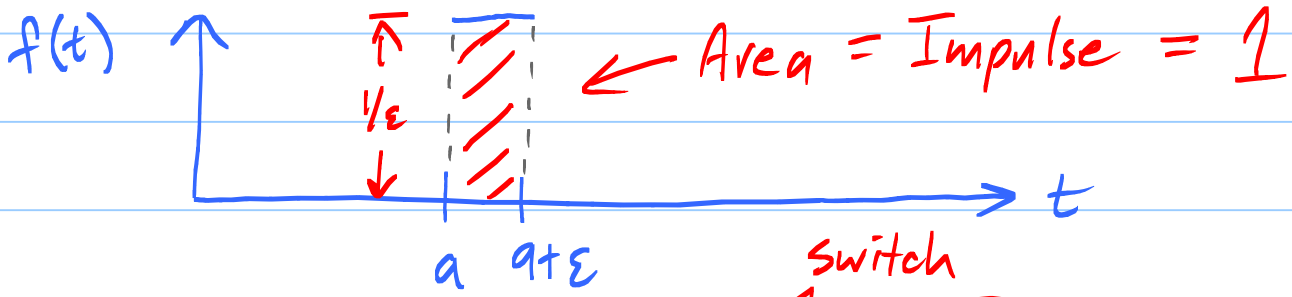
Lesson 21 6.4 18, 19, 20 due Fri, (21, 22 due Wed.)

Skype Hour tonight 7-8 pm purdue.ma527

Impulse fcn (Dirac Delta fcn) $\delta(t)$, $\delta(t-a)$



$$f(t) = \Big[\underbrace{u(t-a) - u(t-(a+\varepsilon))}^{\text{switch}}\Big]\underbrace{\left(\tfrac{1}{\varepsilon}\right)}^{\text{fcn}}$$

$u(t-a)$: on at a

$u(t-(a+\varepsilon))$: off at $a+\varepsilon$

$$F(s) = \frac{1}{\varepsilon}\left(\frac{e^{-as}}{s} - \frac{e^{-(a+\varepsilon)s}}{s}\right)$$

$$= -\frac{1}{s}\left(\underbrace{\frac{e^{-(a+\varepsilon)s} - e^{-as}}{\varepsilon}}_{DQ}\right)$$

$G(a) = e^{-as}$

$G'(a) = -se^{-as}$

$DQ = \dfrac{G(a+\varepsilon) - G(a)}{\varepsilon}$

$\rightarrow G'(a)$ as $\varepsilon \rightarrow 0$.

$$F_\varepsilon(s) \longrightarrow -\frac{1}{s}\left(-se^{-as}\right) = \underline{\underline{e^{-as}}}$$

Think: Perfect sledge hammar hits at time a with unit impulse:

$$\delta(t-a) = \lim_{\varepsilon \to 0} f_\varepsilon(t).$$

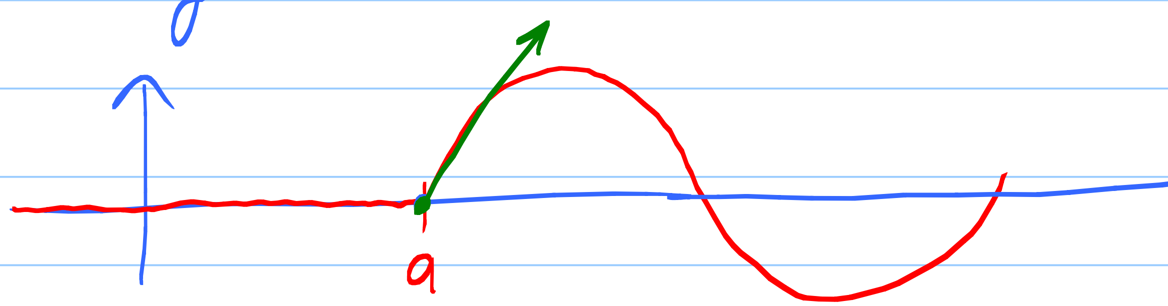
$$\mathcal{L}[\delta(t-a)] = \lim_{\varepsilon \to 0} F_\varepsilon(s) = \underline{\underline{e^{-as}}}$$

Problem: $y'' + y = f_\varepsilon(t)$ $\begin{cases} y(0) = 0 \\ y'(0) = 0 \end{cases}$

$$s^2 Y + Y = F_\varepsilon(s)$$

$$Y = \frac{1}{s^2+1} F_\varepsilon(s) \longrightarrow \frac{e^{-as}}{s^2+1}$$ ← t-shift rule

$$y = u(t-a)\,\sin(t-a)$$



Key properties of $\delta(t-a)$:

1) $\delta(t-a) = 0$ if $t \neq a$
2) Think $\delta(t-a)$ = wham! at time $t=a$.
3) $\int \delta(t-a)\,dt = 1$

4) If h is continuous, then

$$\int h(t)\,\delta(t-a)\,dt = h(a).$$

5) $$\mathcal{L}[\delta(t-a)] = \int_0^\infty \underbrace{e^{-st}}_{h(t)} \delta(t-a)\,dt = \underline{\underline{e^{-as}}}$$

$\delta(t-a)$ is not a true "function." It is a "distribution."

EX: $$y'' + 4y' + 13y = \delta(t-\pi) \qquad \begin{cases} y(0)=0 \\ y'(0)=0 \end{cases}$$

$$\left(s^2 Y - s\underbrace{y(0)}_{0} - \underbrace{y'(0)}_{0}\right) + 4\left(sY - \underbrace{y(0)}_{0}\right) + 13Y = e^{-\pi s}$$

$$Y = \underbrace{\frac{1}{s^2+4s+13}}_{H(s)} \cdot e^{-\pi s}$$ ← t-shifting rule

$$\boxed{y = u(t-\pi)\,h(t-\pi)}$$

$$H(s) = \frac{1}{(s+2)^2+9} = \frac{1}{3}\cdot\frac{3}{(s+2)^2+3^2}$$ s-shifting rule

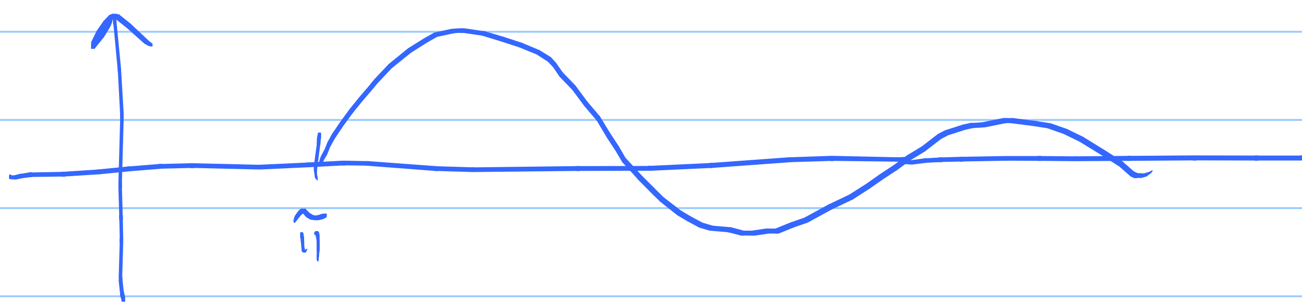
$$\mathcal{L}\left[e^{at} f(t)\right] = F(s-a)$$ $a = -2$

$$h(t) = \frac{1}{3} e^{-2t} \sin 3t$$

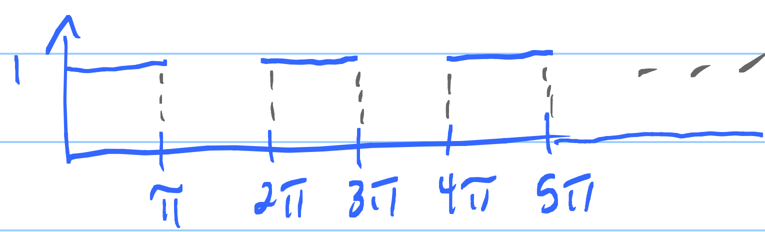
$$y(t) = u(t-\pi)\, h(t-\pi)$$
$$= \underline{u(t-\pi)\, \frac{1}{3}\, e^{-2(t-\pi)} \sin 3(t-\pi)}$$

Note $\quad \sin(3t - 3\pi) = -\sin 3t$

$$e^{-2(t-\pi)} = e^{2\pi}\, e^{-2t}$$



EX: $\quad y'' + y = g(t) =$



$$g(t) = \left[u(t-0) - u(t-\pi)\right] + \left[u(t-2\pi) - u(t-3\pi)\right]$$
$$+ \left[u(t-4\pi) - u(t-5\pi)\right] + \cdots$$

$$G(s) = \left(\frac{1}{s} - \frac{e^{-\pi s}}{s}\right) + \left(\frac{e^{-2\pi s}}{s} - \frac{e^{-3\pi s}}{s}\right) + \cdots$$

$$= \frac{1}{s}\left(1 - e^{-\pi s} + e^{-2\pi s} - e^{-3\pi s} + \cdots\right)$$

Aha! $e^{-n\pi s} = (e^{-\pi s})^n$ Let $r = e^{-\pi s}$ < 1 if $s > 0$.

Geometric Series $1 + r + r^2 + \cdots = \frac{1}{1-r}$

Better: Put $(-r)$ where r is:

$$1 - r + r^2 - r^3 + \cdots = \frac{1}{1-(-r)} = \frac{1}{1+r}$$

$$G(s) = \frac{1}{s}\left[\frac{1}{1+e^{-\pi s}}\right] \leftarrow \text{Laplace Transf of square wave!}$$

Cool, but not as useful as Σ.

$$y'' + y = g(t) \quad \begin{cases} y(0) = 0 \\ y'(0) = 0 \end{cases}$$

$$s^2 Y + Y = \frac{1}{s}\left(1 - e^{-\pi s} + e^{-2\pi s} - \cdots\right)$$

$$Y = \underbrace{\frac{1}{s(s^2+1)}}_{H(s)}\left(1 - e^{-\pi s} + e^{-2\pi s} - \cdots\right)$$

$$Y = H(s) - e^{-\pi s}H(s) + e^{-2\pi s}H(s) - \cdots$$

$$\boxed{y = h(t) - u(t-\pi)h(t-\pi) + u(t-2\pi)h(t-2\pi) - \cdots}$$

$$H(s) = \frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} = \frac{1}{s} - \frac{s}{s^2+1}$$

$$h(t) = \underline{1} - \cos t$$

Note: $\begin{cases} \cos(t-(\text{EVEN})\pi) = \cos t \\ \cos(t-(\text{ODD})\pi) = -\cos t \end{cases}$

Finally, $y(t) = \Big(\overbrace{1 - u(t-\pi) + u(t-2\pi) - \cdots}^{g(t)}\Big)$

$$+ (\cos t)\Big[\underline{1 + u(t-\pi) + u(t-2\pi) + \cdots}\Big]$$

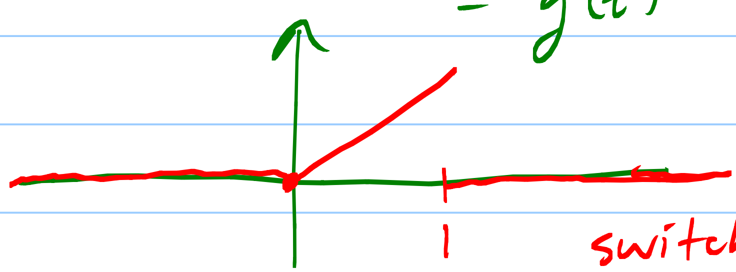
bigger, bigger, bigger, ...

staircase fcn.

Resonance!

223: 25. $y'' + y = \begin{cases} t & 0 < t < 1 \\ 0 & t > 1 \end{cases}$ $\quad y(0) = 0$, $y'(0) = 0$

$= g(t)$



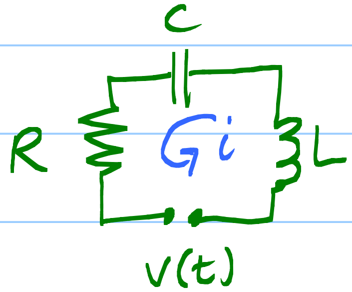
$$g(t) = \Big[\overbrace{u(t-0) - u(t-1)}^{\text{switch fcn}}\Big](t)$$

on at 0 off at 1

$$= \underbrace{u(t-0)(t-0)} - u(t-1)\,\underline{\underline{t}}$$

$$= \quad t \quad - u(t-1)\left[(t-1)+1\right]$$

223: 39.



Kirchoff's Law around loop

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_0^t i(\tilde{\imath})\,d\tilde{\imath} = v(t)$$

Assume $i(0)=0$, charge on capacitor $Q = \int_0^t i(\tilde{\imath})\,d\tilde{\imath}$.

Take Laplace Transform: $L\left(sI(s) - \underbrace{i(0)}_{0}\right) + R\,I(s) + \frac{1}{Cs}I(s) = V(s)$

$$sLI + RI + \frac{1}{Cs}I = \mathcal{L}\left[u(t-0) - u(t-2)\right]$$

$$= \frac{1}{s} - \frac{e^{-2s}}{s}$$

Take it from here.