Lesson 22 6.5 Convolution 21, 22 due Wed.

EX: $y'' + 4y = \sin 2t$ $\begin{cases} y(0) = 0 \\ y'(0) = 0 \end{cases}$

$$s^2 Y + 4Y = \frac{2}{s^2+2^2}$$

$$Y = \frac{2}{(s^2+4)^2} \qquad y = ?$$

Note: Terms like $\frac{As}{(s^2+4)^2}$ and $\frac{B}{(s^2+4)^2}$ happen in partial fractions.

Hmmm: $\cancel{\mathcal{L}[\cos 2t \sin 2t] = \frac{s}{s^2+4} \cdot \frac{1}{s^2+4} = \frac{2s}{(s^2+4)^2}}$

No! $\mathcal{L}[1 \cdot 1] \neq \frac{1}{s} \cdot \frac{1}{s}$

True thing: $\mathcal{L}[f * g] = F(s)\,G(s)$

where $(f*g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$

Discrete version: $\left(\sum_{k=0}^{\infty} a_k z^k\right)\left(\sum_{k=0}^{\infty} b_k z^k\right) = \sum_{n=0}^{\infty} c_n z^n$

where $c_n = \sum_{k=0}^{n} a_k b_{n-k}$.

EX: Find $\mathcal{L}^{-1}\left[\frac{1}{(s^2+1)^2}\right]$.

$$\mathcal{L}[f*g] = \underbrace{\frac{1}{(s^2+1)}}_{F(s)} \cdot \underbrace{\frac{1}{(s^2+1)}}_{G(s)}$$

$f(t) = \sin t \qquad g(t) = \sin t$

Answer: $(f*g)(t) = \int_0^t \sin(\underset{\alpha}{\tau}) \sin(\underset{\beta}{t-\tau})\, d\tau$

Trig: $\sin\alpha \sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$

$$= \frac{1}{2}\int_0^t \cos[\tau-(t-\tau)] - \cos[\tau+(t-\tau)]\, d\tau$$

$$= \frac{1}{2}\left(\int_{\tau=0}^t \cos\underbrace{(2\tau-t)}_{u}\, d\tau - \cos t \underbrace{\int_0^t 1\, d\tau}_{t}\right)$$

$u = 2\tau - t$

$du = 2\,d\tau$

$\boxed{\frac{1}{2}du = d\tau}$

When $\tau = 0$, $u = -t$.

$\tau = t$, $u = 2t - t = t$

$$= \frac{1}{2}\left(\int_{u=-t}^t \cos u \underbrace{\left(\tfrac{1}{2}du\right)}_{d\tau} - t\cos t\right)$$

$$= \frac{1}{2}\left(\frac{1}{2}\left[\sin u\right]_{-t}^{t} - t\cos t\right)$$

$$= \frac{1}{2}\left(\frac{1}{2}\Big(\sin t - \underbrace{\sin(-t)}_{-\sin t}\Big) - t\cos t\right)$$

$$= \frac{1}{2}\sin t - \frac{1}{2}t\cos t$$

Next section 6.6: $\mathcal{L}[t f(t)] = -F'(s)$

Fact: $f * g = g * f$

Why: $\mathcal{L}[f*g] = F(s)\,G(s) = G(s)\,F(s)$
$= \mathcal{L}[g*f]$

So $f*g = g*f$.

EX: $y'' + y = r(t)$ $\quad \begin{cases} y(0)=0 \\ y'(0)=0 \end{cases}$

$$s^2 Y + Y = R(s)$$

$$Y = \underbrace{\frac{1}{s^2+1}}_{\mathcal{L}[\sin t]} \cdot R(s)$$

So $\quad y = (\sin t) * r(t)$

$$= r(t) * \sin t$$

$$\boxed{y = \int_0^t r(\tau) \underbrace{\sin(t-\tau)}_{\text{kernel for this integral operator}} d\tau}$$

kernel for this integral operator

<u>Why</u>: $\mathcal{L}[f*g] = F(s)\,G(s)$

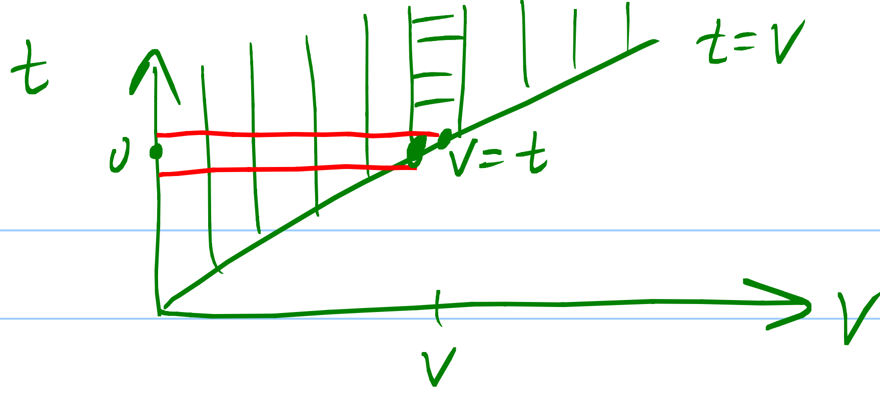
$$F(s)\,G(s) = \left(\int_0^\infty e^{-su} f(u)\,du\right)\left(\int_0^\infty e^{-sv} g(v)\,dv\right)$$

$$= \int_{v=0}^\infty \left(\int_{u=0}^\infty e^{-su} e^{-sv} f(u)\,g(v)\,du\right) dv$$

$$= \int_{v=0}^\infty g(v) \left(\int_{u=0}^\infty e^{-s\underbrace{(u+v)}_{t}} f(u)\,du\right) dv$$

[Let $t = u+v$ ($u = t-v$) When $u=0$, $t=v$
$dt = du$ As $u \to \infty$, so does t

$$= \int_{v=0}^\infty g(v) \left(\int_{t=v}^\infty e^{-st} f(t-v)\,dt\right) dv$$



$$= \int_{t=0}^{\infty} \left(\int_{v=0}^{t} e^{-st} g(v) f(t-v)\, dv \right) dt$$

$$= \int_{t=0}^{\infty} e^{-st} \left(\underbrace{\int_{v=0}^{t} g(v) f(t-v)\, dv}_{g*f \,=\, f*g \ \checkmark} \right) dt$$

<u>Integral Eqns:</u> Volterra Integral Equation of the second kind.

$$y(t) - \int_0^t y(\tau) \sin(t-\tau)\, d\tau = t$$

Hit with $\mathcal{L}$:
$$Y - Y \cdot \frac{1}{s^2+1} = \frac{1}{s^2}$$

$$Y \left[1 - \frac{1}{s^2+1} \right] = \frac{1}{s^2}$$

$$\frac{s^2+1}{s^2+1} - \frac{1}{s^2+1}$$

$$\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$$

$$\mathcal{L}[t^{n-1}] = \frac{(n-1)!}{s^n}$$

$$\mathcal{L}\left[\frac{1}{(n-1)!}\, t^{n-1}\right] = \frac{1}{s^n}$$

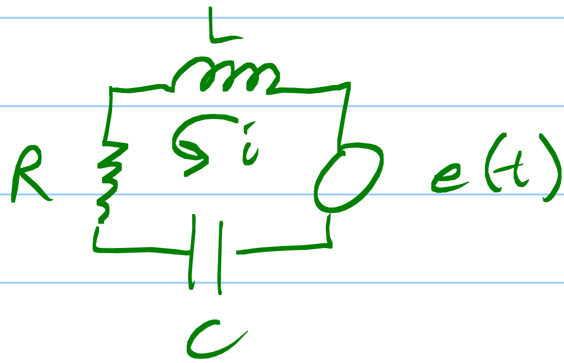
$$Y \cdot \frac{s^2}{s^2+1} = \frac{1}{s^2}$$

$$Y = \frac{s^2+1}{s^4} = \frac{1}{s^2} + \frac{1}{s^4}$$

$$y = t + \frac{1}{3!} t^3$$

$$y = t + \frac{1}{6} t^3$$

Kirchoff's:



$$L\frac{di}{dt} + Ri + \frac{1}{C}Q + \left(-e(t)\right) = 0$$

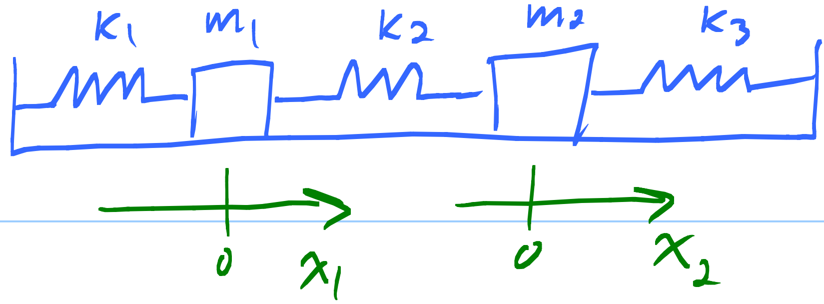
Differentiate, use $i = \frac{dQ}{dt}$, to get 2nd order ODE. What if $e(t)$ = Step function? Ouch! To avoid this, write

$Q = \int_0^t i(\tilde{\tau})\, d\tilde{\tau}$. Use $\mathcal{L}\left[\int_0^t i(\tilde{\tau})\, d\tilde{\tau}\right] = \frac{1}{s} I(s)$.

E.E. comfort: $\frac{d}{dt}\left[u(t-a) - u(t-b)\right] = \delta(t-a) - \delta(t-b)$

"true in the sense of distributions."

Problem:



$$F = ma \qquad \begin{cases} m_1 \ddot{x}_1 = -k_1 x_1 + k_2(x_2 - x_1) \\ m_2 \ddot{x}_2 = -k_3 x_2 - k_2(x_2 - x_1) \end{cases}$$

Take all constants to be equal to 1.

$$\begin{cases} \ddot{x}_1 = -2x_1 + x_2 & \text{(A)} \\ \ddot{x}_2 = x_1 - 2x_2 & \text{(B)} \end{cases}$$

High school method: Solve (B) for x_1:

$$\boxed{x_1 = \ddot{x}_2 + 2x_2}$$

Plug into (A): $\left(\ddot{x}_2 + 2x_2\right)'' = -2\left(\ddot{x}_2 + 2x_2\right) + x_2$

$$x_2^{(4)} + 4\ddot{x}_2 + 3x_2 = 0$$

$$r^4 + 4r^2 + 3 = 0$$

$$(r^2+1)(r^2+3) = 0$$