Lesson 23 6.6 21, 22 due Wed.

$r = \pm i, \pm\sqrt{3}\, i$

$$\begin{cases} x_2 = c_1 \sin t + c_2 \cos t + c_3 \sin\sqrt{3}t + c_4 \cos\sqrt{3}t \\ x_1 = \ddot{x}_2 + 2x_2 = c_1 \sin t + c_2 \cos t - c_3 \sin\sqrt{3}t - c_4 \cos\sqrt{3}t \end{cases}$$

<u>Modes of oscillation</u>: Take one $c = 1$, rest $= 0$.

$x_1 = \sin t$	$x_1 = \cos t$	$x_1 = -\sin\sqrt{3}t$	$x_1 = -\cos\sqrt{3}t$
$x_2 = \sin t$	$x_2 = \cos t$	$x_2 = \sin\sqrt{3}t$	$x_2 = \cos\sqrt{3}t$

<u>EX</u>: Sol^n with $x_1(0) = 1$ $\quad x_2(0) = 0$

$\dot{x}_1(0) = 0 \quad \dot{x}_2(0) = 0$

$$x_1(t) = \frac{1}{2}\left(\cos\sqrt{3}t + \cos t\right) = \left(\cos\frac{\sqrt{3}+1}{2}t\right)\left(\cos\frac{\sqrt{3}-1}{2}t\right)$$

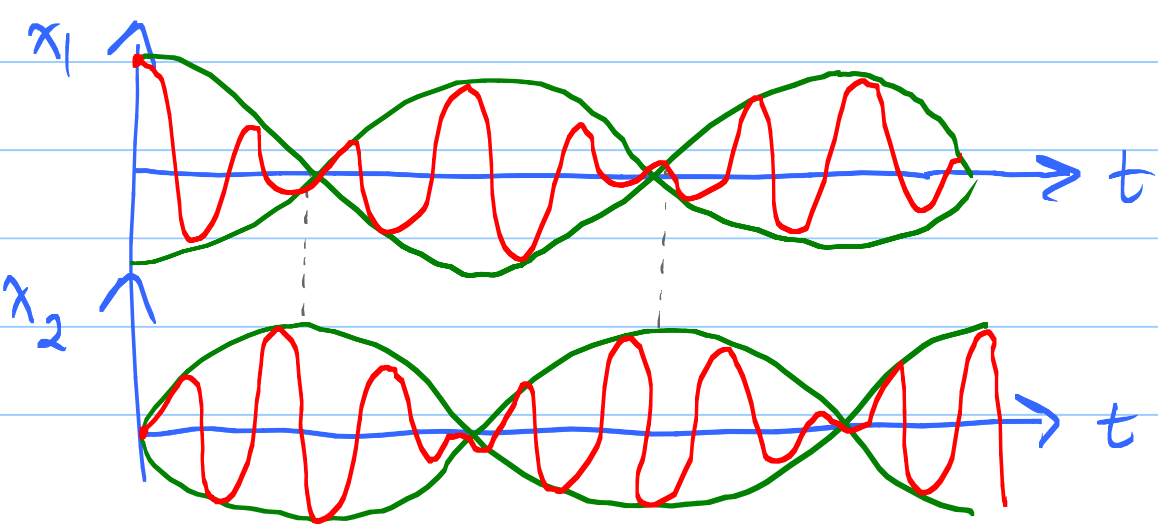
$\cos\alpha + \cos\beta = 2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

$$x_2(t) = \frac{1}{2}\left(-\cos\sqrt{3}t + \cos t\right) = \left(\sin\frac{\sqrt{3}+1}{2}t\right)\left(\sin\frac{\sqrt{3}-1}{2}t\right)$$

$-\cos\alpha + \cos\beta = 2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

Think: $\frac{\sqrt{3}+1}{2}$ fast

$\frac{\sqrt{3}-1}{2}$ slow



6.6 $\mathcal{L}[t f(t)] = -F'(s)$ ← new

$\mathcal{L}[f'(t)] = sF(s) - f(0)$ ← old

Note the symmetry.

EX: $\mathcal{L}[t \underbrace{\sin \omega t}_{f(t)}] = -\frac{d}{ds}\left(\frac{\omega}{s^2+\omega^2}\right)$

$$-\frac{d}{ds}\left[\omega\,(s^2+\omega^2)^{-1}\right] = -\left[-\omega(s^2+\omega^2)^{-2}(2s)\right]$$

$$= \frac{2\omega s}{(s^2+\omega^2)^2}$$

EX: $\mathcal{L}[t \cos \omega t] = \frac{s^2-\omega^2}{(s^2+\omega^2)^2} = \frac{s^2+\omega^2-\omega^2-\omega^2}{(s^2+\omega^2)^2}$

$$= \frac{1}{\omega} \cdot \frac{\omega}{(s^2+\omega^2)} - \frac{2\omega^2}{(s^2+\omega^2)^2}$$

$f(t)$	$F(s)$
$\frac{1}{2\omega^2}(\sin \omega t - \omega t \cos \omega t)$	$\frac{1}{(s^2+\omega^2)^2}$
$\frac{1}{2\omega} t \sin \omega t$	$\frac{s}{(s^2+\omega^2)^2}$

Why: $F'(s) = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt$

$$= \int_0^\infty \frac{d}{ds}\left[e^{-st}\right] f(t)\,dt$$

$$= \int_0^\infty \left[-t e^{-st}\right] f(t)\,dt$$

$$= -\mathcal{L}\left[t f(t)\right] \checkmark$$

Fact: If $f(0)=0$ and $\lim_{t\to 0+} \frac{f(t)}{t}$ exists,

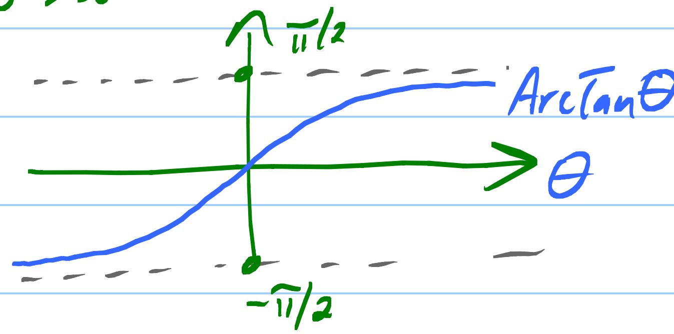
then $\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds$

Note: $\frac{f(t)}{t} = \frac{f(t) - \overset{0}{f(0)}}{t - 0} \rightarrow$ Right hand derivative of f at $t=0$ as $t \to 0+$

EX: $\mathcal{L}\left[\frac{\sin t}{t}\right] = \int_s^\infty F(s)\,ds = \int_s^\infty \frac{1}{s^2+1}\,ds$ ← $f(t) = \sin t$

$$= \left[\operatorname{ArcTan} s\right]_s^{\infty} = \lim_{B\to\infty}\left[\operatorname{ArcTan} B - \operatorname{ArcTan} s\right]$$

$$= \frac{\pi}{2} - \operatorname{ArcTan} s$$



Check: $f(t) = \sin t$

$f(0) = 0$ ✓

$\sin t$ has a derivative at $t=0$. So R.H. derivative exists. $\lim_{t\to 0+} \frac{\sin t}{t}$ exists. ✓

Special ODEs with variable coefficients.

Laguerre Eqn: $ty'' + (1-t)y' + ny = 0$

$$\mathcal{L}[ty'] = -\frac{d}{ds}\mathcal{L}[y'] = -\frac{d}{ds}\left[sY - y(0)\right]$$

$$= -Y - sY' + 0 \quad (*_1)$$

$$\mathcal{L}[ty''] = -\frac{d}{ds}\mathcal{L}[y''] = -\frac{d}{ds}\left[s^2 Y - sy(0) - y'(0)\right]$$

$$= -2sY - s^2 Y' + y(0) \quad (*_2)$$

$\mathcal{L}[ODE]$: $\quad ty'' + y' - ty' + ny = 0$

$$\left(-2sY - s^2 Y' + y(0)\right) + \left(sY - y(0)\right) - \left(-Y - sY'\right) + nY = 0$$

cancel

$$(s - s^2) Y' + (n+1-s) Y = 0$$

First order ODE from a 2nd order one!

Separable!

$$\boxed{\frac{dY}{ds} = -\frac{(n+1-s)}{s(1-s)} Y}$$

$$\frac{dY}{Y} = -\frac{(n+1-s)}{s(1-s)} ds$$

Integrate: $\quad \displaystyle\int \frac{dY}{Y} = \int \frac{-(n+1-s)}{\underbrace{s(1-s)}} ds$

$$\frac{A}{s} + \frac{B}{1-s} = \frac{n}{s-1} - \frac{n+1}{s}$$

$$\ln|Y| = \int \frac{n}{s-1} - \frac{n+1}{s} \, ds$$

$$= n \ln(s-1) - (n+1)\ln s + C$$

exponentiate: $\quad |Y| = e^{\ln(s-1)^n} \cdot e^{-\ln s^{n+1}} \cdot e^{C}$

$$Y = \underbrace{\pm e^{C}}_{K} (s-1)^n \frac{1}{s^{n+1}}$$

$$\bar{Y} = K \frac{(s-1)^n}{s^{n+1}}$$

Take $k=1,\ n=3$: $\bar{Y} = \frac{(s-1)^3}{s^4} = \frac{1}{s} - \frac{3}{s^2} + \frac{3}{s^3} - \frac{1}{s^4}$

$$y = 1 - 3t + \frac{3}{2}t^2 - \frac{1}{6}t^3$$