Lesson 24 6.7, 12.12 HWK 8 23, 24, 25 due Wed.

Laguerre Eqn $n=3$: $t y'' + (1-t) y' + 3y = 0$ (*)

If a nice sol^n to (*) exists, we can use $\mathcal{L}$ to find it. Got $y_1(t) = 1 - 3t + \frac{3}{2}t^2 - \frac{1}{6}t^3$ (Laguerre polynomial.) It is a sol^n. Check it.

Laguerre Eqn Standard Form: $y'' + \overbrace{\left(\frac{1}{t} - 1\right)}^{P} y' + \overbrace{\frac{3}{t}}^{Q} y = 0$

ouch at $t=0$

Fact: P, Q good $\Rightarrow$ sol^n good.

Reduction of order: $y_2 = u y_1$ where $u' = \frac{1}{y_1^2} e^{-\int P\,dt}$

$$u' = \frac{1}{y_1^2} e^{-\int \frac{1}{t} - 1\,dt} = \frac{1}{t y_1^2} e^t$$

u has a logarithmic singularity at $t=0$.

So y_2 is singular at $t=0$ too. No wonder $\mathcal{L}$ missed it!

Using $\mathcal{L}$ to solve systems: $\begin{cases} \ddot{x}_1 = -2x_1 + x_2 \\ \ddot{x}_2 = x_1 - 2x_2 \end{cases}$

I.C. $\begin{cases} x_1(0) = 1, & \dot{x}_1(0) = 0 \\ x_2(0) = 0, & \dot{x}_2(0) = 0 \end{cases}$

Hit system with $\mathcal{L}$:

$$\begin{cases} s^2 X_1 - s \cdot \overset{1}{x_1(0)} - \overset{0}{\dot{x}_1(0)} = -2X_1 + X_2 \\ s^2 X_2 - s\, \overset{0}{x_2(0)} - \overset{0}{\dot{x}_2(0)} = X_1 - 2X_2 \end{cases}$$

$$\begin{cases} -s = (-2-s^2)X_1 + X_2 \\ 0 = X_1 + (-2-s^2)X_2 \end{cases}$$

$$\begin{pmatrix} -s \\ 0 \end{pmatrix} = \begin{bmatrix} -2-s^2 & 1 \\ 1 & -2-s^2 \end{bmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

Cramer's:

$$X_1 = \frac{\det \begin{bmatrix} -s & 1 \\ 0 & -2-s^2 \end{bmatrix}}{\det \begin{bmatrix} -2-s^2 & 1 \\ 1 & -2-s^2 \end{bmatrix}}$$

$$= \frac{s(s^2+2)}{(s^2+2)^2 - 1} = \frac{s^3+2s}{s^4+4s^2+3}$$

$$= \frac{s^3+2s}{(s^2+1)(s^2+3)}$$

$$= \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+3}$$

$\mathcal{L}$ of linear combo of $\sin t, \cos t$, $\sin\sqrt{3}t, \cos\sqrt{3}t$

$$X_2 = \frac{\det \begin{bmatrix} -2-s^2 & -s \\ 1 & 0 \end{bmatrix}}{s^4+4s^2+3} = \frac{s}{(s^2+1)(s^2+3)} \quad \ldots \text{ etc.}$$

College way:

$$\ddot{x}_1 = -2x_1 + x_2 \quad (A)$$

$$\ddot{x}_2 = x_1 - 2x_2 \quad (B)$$

$$\begin{cases} u_1 = x_1 \\ u_2 = \dot{x}_1 \\ u_3 = x_2 \\ u_4 = \dot{x}_2 \end{cases}$$

$$\begin{cases} u_1' = \dot{x}_1 = u_2 \\ u_2' = \ddot{x}_1 \overset{(A)}{=} -2x_1 + x_2 = -2u_1 + u_3 \\ u_3' = \dot{x}_2 = u_4 \\ u_4' = \ddot{x}_2 \overset{(B)}{=} x_1 - 2x_2 = u_1 - 2u_3 \end{cases}$$

$$\vec{u}\,' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & -2 & 0 \end{bmatrix} \vec{u}$$

Get $(\lambda^2+1)(\lambda^2+3) = 0$ e-vals $\pm i$, $\pm\sqrt{3}\,i$

For $\lambda = i$, get complex e-vect. $\begin{pmatrix} 1 \\ i \\ 1 \\ i \end{pmatrix} = \vec{a} + \vec{b}i$

where $\vec{a} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, $\vec{b} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$.

Get two real sol^{n}s from $(\vec{a} + \vec{b}i)e^{it}$.

For $\lambda = \sqrt{3}\,i$, get complex $\vec{A} + \vec{B}i$ e-vect.

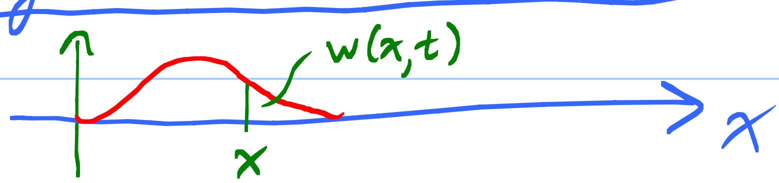
Get two real sol^{n}s from $(\vec{A} + \vec{B}i)e^{\sqrt{3}\,it}$

$$\vec{u} = c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 + c_4\vec{u}_4 = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix} \begin{array}{l} \leftarrow x_1 \\ \\ \leftarrow x_2 \\ \\ \end{array}$$

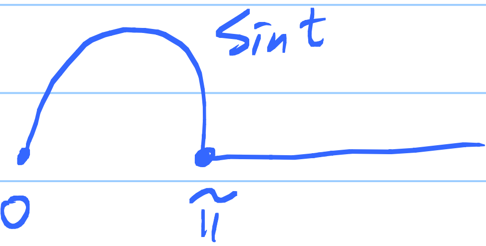
12.12 Using $\mathcal{L}$ to solve a PDE:

Whip a rope

Prob:



Wave eqn: $\dfrac{\partial^2 w}{\partial t^2} = c^2 \dfrac{\partial^2 w}{\partial x^2}$ $c =$ speed of propagation

$w(\underset{=}{0}, t) =$  $= [u(t-0) - u(t-\pi)] \sin t$

left end

Energy considerations: $\lim_{x\to\infty} w(x, t) = 0$ ↑ t held fixed

Initial Conditions: $\begin{cases} w(x,0) = 0 \\ \dfrac{\partial w}{\partial t}(x,0) = 0 \end{cases}$ "dead rope"

Hit PDE with $\mathcal{L}$ in t-variable,

$$\mathcal{L}[w(x,t)] = \int_0^\infty e^{-st} w(x,t)\,dt$$

$$= W(x, s)$$

$\mathcal{L}$: $\dfrac{\partial^2 w}{\partial t^2} = c^2 \dfrac{\partial^2 w}{\partial x^2}$

$$s^2 W - \underbrace{w(x,0)}_{0} s - \underbrace{\frac{\partial w}{\partial t}(x,0)}_{0} =$$

$$= c^2 \int_0^\infty e^{-st} \frac{\partial^2 w}{\partial x^2}(x,t)\,dt$$

$$= c^2 \frac{\partial^2}{\partial x^2} \int_0^\infty e^{-st} w(x,t)\,dt$$

$$= c^2 \frac{\partial^2}{\partial x^2} \overline{W}(x,s)$$

$$\boxed{s^2 \overline{W} = c^2 \frac{\partial^2}{\partial x^2} \overline{W}}$$

treat like an ODE in x with parameter s floating along.

$$\frac{\partial^2 \overline{W}}{\partial x^2} - \frac{s^2}{c^2} \overline{W} = 0$$

$$y'' - k^2 y = 0 \qquad y = c_1 e^{kx} + c_2 e^{-kx}$$

$$\overline{W} = c_1(s) e^{sx/c} + c_2(s) e^{-sx/c}$$

[Note: Any reasonable Laplace trans $\to 0$ as $s \to \infty$. Suspect that $c_1(s) = 0$.

Energy considerations: $\overline{W}(x,s) = \int_0^\infty e^{-st} \underbrace{w(x,t)}_{\to 0 \text{ as } x \to \infty} dt$

$\overline{W}(x,s) \to 0$ as $x \to \infty$. Aha! $c_1(s)$ must be 0.

So $c_1(s) \equiv 0$. Get $\boxed{\overline{W}(x,s) = c_2(s) e^{-sx/c}}$

What is $c_2(s)$? $\overline{W}(0,s) = \int_0^\infty e^{-st} \underbrace{w(0,t)}_{\text{one hump of Sin}} dt$

$$= F(s) \leftarrow c_2(s)$$

So $\bar{W}(x,s) = F(s)\, e^{-sx/c}$.

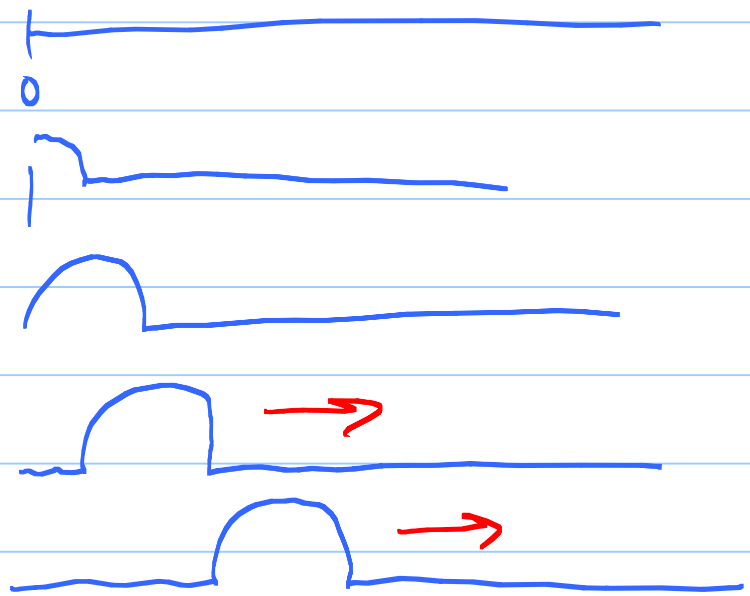
Undo $\mathcal{L}$:

$$w(x,t) = \mathcal{L}^{-1}\left[F(s)\, e^{-sx/c}\right]$$

$$= u\left(t-\frac{x}{c}\right) f\left(t-\frac{x}{c}\right)$$

one hump of sin

$t=0$



MAPLE: plot(Heaviside(t-Pi)*sin(t-Pi),
t = 0..35);

?animate