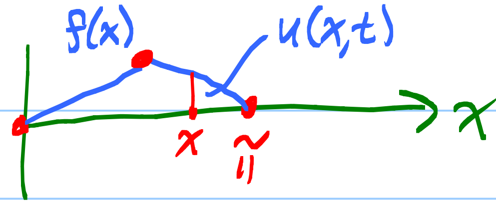
# Lesson 25 11.1 Fourier Series

HWK 23, 24, 25 due Wed

## Harpsichord Prob:




$$c^2 \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial t^2} \quad (*)$$

(Take $c=1$.)

I.C. (Initial Cond.) $\begin{cases} u(x,0) = f(x) \\ \frac{\partial y}{\partial t}(x,0) = 0 \end{cases}$

B.C. (Boundary Cond.) $\begin{cases} u(0,t) = 0 \\ u(\pi,t) = 0 \end{cases}$

Bernoulli v.s. Bernoulli: Jacob: $u(x,t) = \varphi(x+t) - \psi(x-t)$

Johann: Try $u(x,t) = X(x)\,T(t)$

Plug into $(*)$ $\frac{\partial^2 y}{\partial x^2} = X''(x)T(t) \overset{\text{want}}{=} X(x)T''(t) = \frac{\partial^2 y}{\partial t^2}$

$$\frac{X''}{X} = \frac{T''}{T}$$

Hmmm. Plug in $x = \frac{\pi}{2}$.
See $\frac{T''}{T}$ must be const!
Call it $\lambda$. $\frac{X''}{X} \equiv \lambda$ too.

Get two ODEs. $\frac{X''}{X} = \lambda$, $\frac{T''}{T} = \lambda$

Save Initial Conditions for later.

Boundary Conditions: $u(0,t) = X(0)T(t) \overset{\text{need}}{=} 0$

$X(0) = 0$

$u(\pi,t) = X(\pi)T(t) = 0$

need $X(\pi) = 0$

$$T'' - \lambda T = 0$$

$$\begin{cases} X'' - \lambda X = 0 \\ X(0) = 0,\ X(\pi) = 0 \end{cases}$$

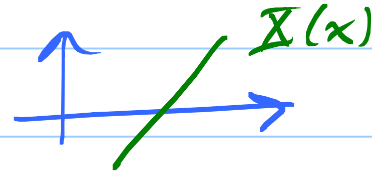
Sturm-Liouville Prob.

Think: $L = \frac{d^2}{dx^2}$

ODE: $LX = \lambda X$

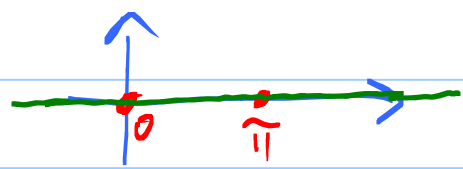
$\lambda$ e-val for $L$!

Vector space: $C^2$-fcns vanishing at $0$ and $\pi$.




Case $\lambda = 0$: $X'' = 0$

$X(x) = c_1 x + c_2$




Ouch! Only get $X(x) \equiv 0$.

Case $\lambda > 0$: Write $\lambda = k^2$. $X'' - k^2 X = 0$

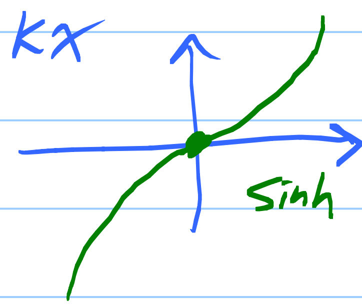
$r^2 - k^2 = 0,\ r = \pm k$

$X(x) = c_1 e^{kx} + c_2 e^{-kx}$

B.C. Need $X(0) = c_1 e^{k\cdot 0} + c_2 e^{-k\cdot 0} \overset{\text{want}}{=} 0$

$c_1 + c_2 = 0$. $\boxed{c_2 = -c_1}$

So $X(x) = c_1 e^{kx} - c_1 e^{-kx} = 2c_1 \sinh kx$




Also need $X(\pi) = 2c_1 \underbrace{\sinh k\pi}_{\neq 0}$

Ouch! $c_1$ must $= 0$.

Darn! Only $X(x) \equiv 0$ here too.

Case $\lambda < 0$: Write $\lambda = -\mu^2$. $X'' - \lambda X = 0$

$X'' + \mu^2 X = 0$

$r^2 + \mu^2 = 0,\ r = \pm \mu i$

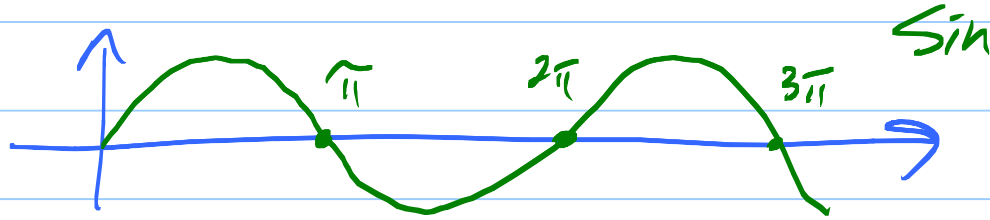
$X(x) = c_1 \sin \mu x + c_2 \cos \mu x$

B.C. Need $X(0) = c_1 \sin 0 + c_2 \cos 0 = c_1 \cdot 0 + c_2 \cdot 1$

$= c_2 \overset{\text{want}}{=} 0$ $\quad \boxed{c_2 = 0}$

So $X(x) = c_1 \sin \mu x$

Also need $X(\pi) = c_1 \sin \mu \pi \overset{\text{want}}{=} 0$




Aha! Must have $\mu \pi$ be a multiple of $\pi$,

i.e., $\mu \pi = n \pi$, $n = 1, 2, 3, \ldots$

E-vals! $\boxed{\mu = n}$ $\quad \underline{\lambda = -n^2}$

Get non-zero sol^ns $X_n(x) = \sin nx$ $\left(\text{Take } c_1 = 1\right)$

Get $T_n(t) = c_1 \cos nt + c_2 \sin nt$

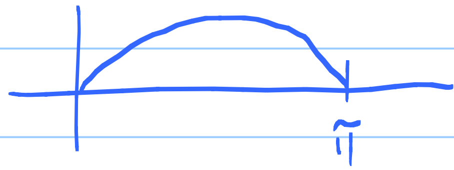
Initial Cond: $\frac{\partial u}{\partial t}(x, 0) \equiv 0$ $\quad$ need $c_2 = 0$

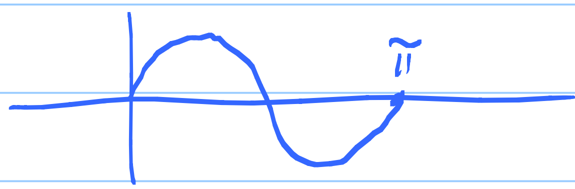
$T_n'(0) = -n c_1 \sin 0 + n c_2 \cos 0 = n c_2$

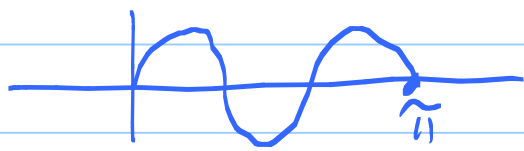
Take $c_1 = 1$ too for now. So $T_n(t) = \cos nt$.

Found $\boxed{u_n(x, t) = \sin nx \cos nt}$

$$X_n(x) = \sin nx$$

$n=1$  $\sin x$ | $\cos t$

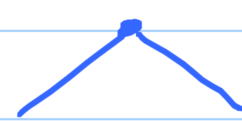
$n=2$  $\sin 2x$ | $\cos 2t$

$n=3$  $\sin 3x$ | $\cos 3t$

PDE is linear! B.C. are homogeneous. Can add sol$^n$s to get more sol$^n$s! Try

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin nx \cos nt$$

Initial Cond.? $\quad u(x,0) = \sum_{n=1}^{\infty} c_n \sin nx$

Question: Can $\sum_{n=1}^{\infty} c_n \sin nx =$  ?

Yes! $\sin nx$ are orthogonal on $[0,\pi]$.

$$(\sin nx, \sin mx) = \int_0^{\pi} \sin nx \sin mx \, dx = \begin{cases} 0 & n \neq m \\ \frac{\pi}{2} & n = m \end{cases}$$

$$\left[ \sin^2\theta = \tfrac{1}{2}(1 - \cos 2\theta) \right]$$

Big idea: Want $\displaystyle\sum_{1}^{\infty} c_n \sin nx = f(x)$

Multiply by $\sin mx$: $\displaystyle\sum_{n=1}^{\infty} c_n \sin nx \sin mx = f(x) \sin mx$

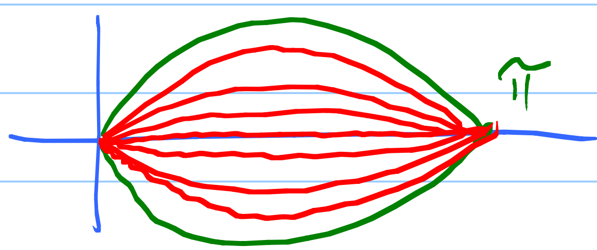
Integrate $\int_0^{\pi}$: $\displaystyle\sum_{n=1}^{\infty} c_n \underbrace{\int_0^{\pi} \sin nx \sin mx \, dx}_{\text{all } = 0 \text{ except } n=m \text{ one!}} = \int_0^{\pi} f(x) \sin mx \, dx$

$$c_m \underbrace{\int_0^{\pi} \sin^2 mx \, dx}_{\frac{\pi}{2}} = \int_0^{\pi} f(x) \sin mx \, dx$$

$$\boxed{c_m = \frac{2}{\pi} \int_0^{\pi} f(x) \sin mx \, dx}$$

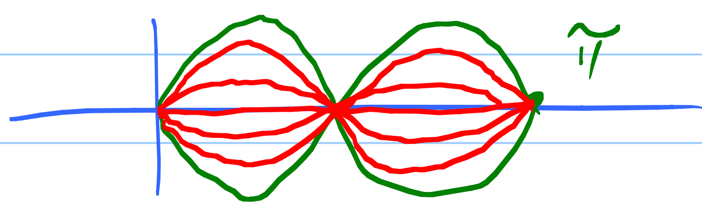
$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin nx \cos nt$$

$n=1$



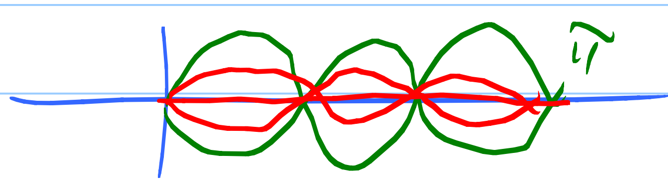

$\sin x \cos t$

$n=2$




$\sin 2x \cos 2t$

$n=3$




$\sin 3x \cos 3t$

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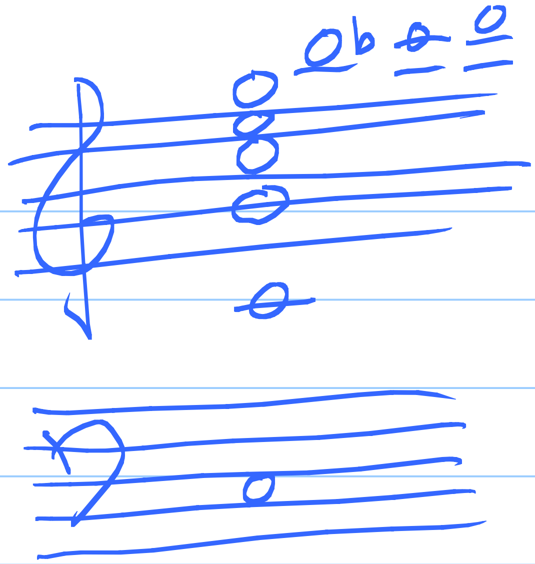

⋮
n=4
n=3
n=2

n=1