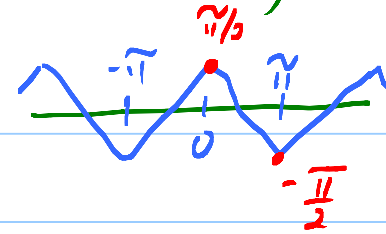
Lesson 27 11.3, 11.4 (26, 27, 28 due Wed.)

11.3 $\underset{\substack{\uparrow \\ m=1}}{y''} + \underset{\substack{\uparrow \\ c \text{ due to friction}}}{0.05}\, y' + \underset{\substack{\uparrow \\ k}}{25}\, y = F(x)$

Spring mass



Frictionless homog prob: $y'' + 25y = 0$

$y = c_1 \cos 5x + c_2 \sin 5x$

↑natural freq

Friction (homog prob) $y'' + .05y' + 25y = 0$

$r^2 + .05r + 25 = 0$

$$r = \frac{-.05}{2} \pm \frac{\sqrt{(.05)^2 - 100}}{2} = \begin{pmatrix}\text{neg.}\\\text{small}\end{pmatrix} \pm \begin{pmatrix}\text{close}\\\text{to}\\5\end{pmatrix} i = -a \pm bi$$

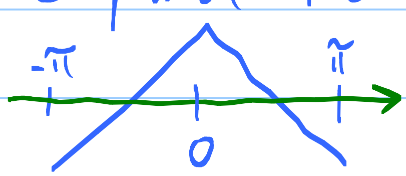
$$y = c_1 e^{-ax}\cos bx + c_2 e^{-ax}\sin bx$$

friction causes this homog sol^n to die away.

This solution is transient.

Back to non-homog prob: Need particular sol^n.

Plan: Expand $F(x)$ in a Fourier Series.



$a_0,\ a_n,\ b_n = \ldots$

$$F(x) = \frac{4}{\pi}\left(\cos x + \frac{1}{3^2}\cos 3x + \frac{1}{5^2}\cos 5x + \cdots\right)$$

$$= \frac{4}{\pi}\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2}\cos(2n+1)x$$

$x=0$: $F(0) = \frac{\pi}{2} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

Plan: Get part. sol^n y_{p_n} for RHS

$F_n(x) = \frac{4}{\pi n^2}\cos nx \leftarrow n$ odd

Then $y_p = y_{p_1} + y_{p_3} + y_{p_5} + \cdots$

Prob: $y'' + .05y' + 25y = \frac{4}{\pi n^2}\cos nx$

Undet. Coeff: Try y_{p_n} of the form

safe!

$(25)\times \quad y_{p_n} = A_n\cos nx + B_n \sin nx$

$(.05)\times \quad y_{p_n}' = -nA_n\sin nx + nB_n\cos nx$

$(1)\times \quad y_{p_n}'' = -n^2A_n\cos nx - n^2B_n\sin nx$

want

$$(\underbrace{-n^2A_n + .05nB_n + 25A_n}_{\frac{4}{\pi n^2}})\cos nx + (\underbrace{-n^2B_n - .05nA_n + 25B_n}_{0})\sin nx \overset{\text{want}}{=} \frac{4}{\pi n^2}\cos nx + 0\cdot\sin nx$$

$$\begin{cases} (25-n^2)A_n + .05n B_n = \dfrac{4}{\pi n^2} \\ -.05\, n A_n + (25-n^2) B_n = 0 \end{cases}$$

Cramer's:

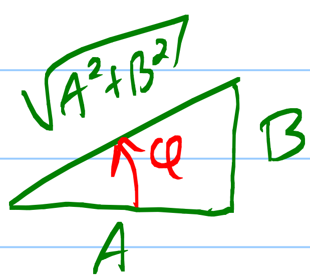
$$A_n = \frac{4(25-n^2)}{\pi n^2 \left[(25-n^2)^2 + (.05n)^2\right]}$$

$$B_n = \frac{.2}{\pi n \left[(25-n^2)^2 + (.05n)^2\right]}$$

Something big happens when $n=5$!

Important trick: $A \cos \omega t + B \sin \omega t =$

$$\sqrt{A^2+B^2}\left(\underbrace{\frac{A}{\sqrt{A^2+B^2}}}_{\cos\varphi} \cos \omega t \quad \underbrace{\frac{B}{\sqrt{A^2+B^2}}}_{\sin\varphi} \sin \omega t\right) =$$



$$= \sqrt{A^2+B^2} \cos(\omega t - \varphi)$$

Back to y_{P_n}:

$$y_{P_n} = \sqrt{A_n^2 + B_n^2} \cos(nx - \varphi_n)$$

$$\text{Amplitude} = \frac{4}{n^2 \pi^2 \sqrt{(25-n^2)^2 + (.05n)^2}}$$

Gigantic when $n=5$. Small for rest.
Fizzling out for big n.

Think: $n=5$ is close to natural freq of spring mass. Sympathetic vibrations!

11.4: Orthogonal expansions.

$L^2[-\pi,\pi]$ = integrable functions f such that

$$\int_{-\pi}^{\pi} f^2\,dx < \infty.$$

"Square integrable functions".

$$f = \sum_{n=1}^{\infty} c_n \varphi_n \quad \text{where } \varphi_n \text{ orthogonal.}$$

$$(\varphi_n, \varphi_m) = \int_{-\pi}^{\pi} \varphi_n(x)\varphi_m(x)\,dx = 0 \text{ if } n \neq m.$$

Normalize: Divide φ_n by $\sqrt{\int_{-\pi}^{\pi} \varphi_n^2\,dx}$ so

we get orthonormal φ_n: $\|\varphi_n\| = \sqrt{\int_{-\pi}^{\pi} \varphi_n^2\,dx} = 1$

Suppose $f = \sum_{n=1}^{\infty} c_n \varphi_n$

Mult by φ_m: $\quad f\varphi_m = \sum_{n=1}^{\infty} c_n \varphi_n \varphi_m$

Integrate: $\int_{-\pi}^{\pi} f\varphi_m\,dx = \sum_{n=1}^{\infty} c_n \underbrace{\int_{-\pi}^{\pi} \varphi_n \varphi_m\,dx}$

all $=0$ except $n=m$ term, which is one

$$\boxed{\int_{-\pi}^{\pi} f\,\varphi_m\,dx = c_m}$$

Fact: Among all linear combo's $\sum_{n=1}^{N} A_n \varphi_n$, the choice of $A_n = c_n$, $n=1,\ldots,N$ makes $\left\| f - \sum_{n=1}^{N} A_n\varphi_n \right\|$ an absolute minimum.

Why:

$$\overset{E}{E^*} = \left\| f - \sum_{n=1}^{N} \overset{A_n}{c_n}\varphi_n \right\|^2 =$$

$$\int_{-\pi}^{\pi} \left(f - \sum_{n=1}^{N} \overset{A_n}{c_n}\varphi_n\right)\left(f - \sum_{m=1}^{N} \overset{A_m}{c_m}\varphi_m\right)dx$$

$$= \int_{-\pi}^{\pi} f^2\,dx - \sum_{n=1}^{N} \overset{A_n}{c_n} \underbrace{\int_{-\pi}^{\pi} \varphi_n f\,dx}_{c_n} - \sum_{m=1}^{N} \overset{A_m}{c_m} \underbrace{\int_{-\pi}^{\pi} \varphi_m f\,dx}_{c_n}$$

$$+ \sum_{n=1}^{N}\sum_{m=1}^{N} \underset{A_n}{c_n}\,\underset{A_m}{c_m} \underbrace{\int_{-\pi}^{\pi} \varphi_n\varphi_m\,dx}_{\delta_{nm}}$$

$$= \int_{-\pi}^{\pi} f^2\,dx - \sum_1^N c_n^2 - \sum_1^N c_n^2 + \sum_1^N c_n^2$$

$$= \int_{-\pi}^{\pi} f^2\,dx - \sum_{n=1}^N c_n^2$$

$$0 \le E^* = \int_{-\pi}^{\pi} f^2\,dx - \sum_{n=1}^N c_n^2$$

$$\sum_{n=1}^N c_n^2 \le \int_{-\pi}^{\pi} f^2\,dx$$

Bessel's Ineq. Let $N \to \infty$!

Get $E - E^* = \sum_{n=1}^N (A_n - c_n)^2$ ← sum of squares

≥ 0 ← only equal to zero if $A_n = c_n$ $n=1,\dots,N$!

Do this same routine for Fourier Series.

Bessel's Ineq: $\quad 2a_0^2 + \sum_{n=1}^\infty (a_n^2 + b_n^2) \le \frac{1}{\pi}\int_{-\pi}^{\pi} f^2\,dx$

Parseval's Identity: $=$

Word: $a_0 + \sum_{n=1}^N (a_n \cos nx + b_n \sin nx)$ ← Trigometric Poly