Lesson 28 11.5 Sturm-Liouville Probs. (26, 27, 28 due Wed.)

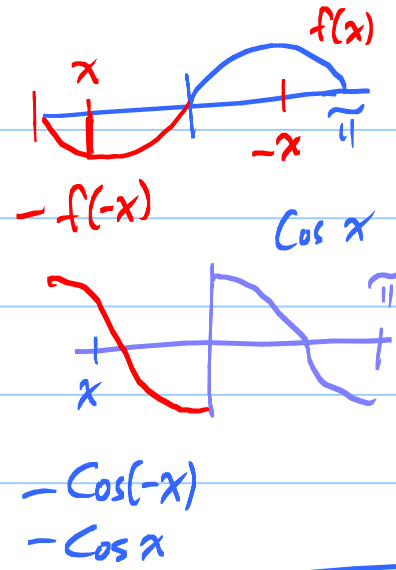
EX: $y'' - \lambda y = 0$ ← parameter

$\underbrace{y(0)=0, \quad y(\pi)=0}$

B.C. Boundary Conditions

λ eigenvalues, Non-zero

sol^{n}s for the special λ evals are called eigenfuctions.



Sin x
$c_1 = 1$
all other's $= 0$

Big Fact: The e-fcns are orthogonal.

Similar to symmetric matrix fact. $\begin{cases} A\vec{x}_1 = \lambda_1 \vec{x}_1 \\ A\vec{x}_2 = \lambda_2 \vec{x}_2 \end{cases}$

$$\lambda_1 (\vec{x}_1, \vec{x}_2) = (\lambda_1 \vec{x}_1, \vec{x}_2)$$
$$= (A\vec{x}_1, \vec{x}_2) = (\vec{x}_1, A^T \vec{x}_2) = (\vec{x}_1, A\vec{x}_2)$$
$$A^T = A$$
$$= (\vec{x}_1, \lambda_2 \vec{x}_2) = \lambda_2 (\vec{x}_1, \vec{x}_2)$$

$$\underbrace{(\lambda_1 - \lambda_2)}_{\text{not zero}} \underbrace{(\vec{x}_1, \vec{x}_2)}_{\text{must be zero}} = 0$$

$y'' = \lambda y$

$y(0) = 0, \quad y(\pi) = 0$

Two eigenfunctions y_1, y_2 for $\lambda_1 \neq \lambda_2$.

$$\lambda_1 (y_1, y_2) = \int_0^{\pi} \underbrace{\lambda_1 y_1}_{y_1''} \, y_2 \, dx$$

$$= \int_0^{\pi} \underset{u}{y_2} \, \underbrace{y_1'' \, dx}_{dv}$$

$$\left[\begin{array}{ll} u = y_2 & du = y_2' \, dx \\ dv = y_1'' \, dx & v = y_1' \end{array} \right.$$

$$= uv \Big|_0^{\pi} - \int_0^{\pi} v \, du$$

$$= \underbrace{y_2 y_1' \Big|_0^{\pi}}_{\text{zero because of B.C.!}} - \int_0^{\pi} y_1' y_2' \, dx$$

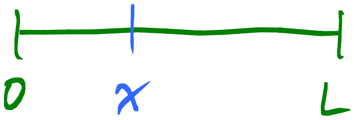
Aha! Repeat: $\lambda_2 (y_1, y_2) = \lambda_2 (y_2, y_1) = \cdots$

$$= - \int_0^{\pi} y_2' y_1' \, dx$$

$$\boxed{\lambda_1 (y_1, y_2) = \lambda_2 (y_1, y_2)}$$

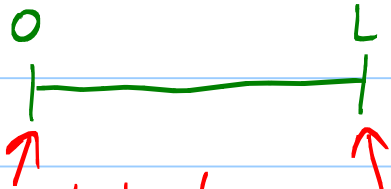
$$\underbrace{(\lambda_1 - \lambda_2)}_{\text{not zero}} \underset{\text{must be zero}}{(y_1, y_2)} = 0$$

Heat problem: Hot wire



$u(x,t)$ = temp of wire at x at time t.

Heat Eqn: $\dfrac{\partial^2 u}{\partial x^2} = c\,\dfrac{\partial u}{\partial t}$



let radiate heat

$\dfrac{\partial u}{\partial x}(0,t) = -k\,u(0,t)$

ice this end

$u(L,t) = 0$

Take c and k both one.

Try $u(x,t) = X(x)\,T(t)$.

PDE: $X''T = XT'$

$$\frac{X''}{X} = \frac{T'}{T}$$ ← both const., call it λ.

$X'' - \lambda X = 0$ | $T' - \lambda T = 0$ ← easy!

$T(t) = ce^{-\lambda t}$

B.C.: 1) $u(L,t) = 0$

$X(L)\,T(t) = 0$

don't want $T(t) \equiv 0$.

Must have $\underline{\underline{X(L)=0}}$.

2) $\frac{\partial y}{\partial x}(0,t) + u(0,t) = 0$ for all t.

$$X'(0)T(t) + X(0)T(t) = 0$$

$$\underbrace{[X'(0)+X(0)]}_{\text{must be zero}}T(t) = 0$$

Sturm-Liouville: $X'' - \lambda X = 0 \leftarrow$ ODE

$$\text{B.C.} \begin{cases} X'(0)+X(0)=0 \\ X(L)=0 \end{cases}$$

Sol^n:

$$X(x) = c_1 x + c_2 \qquad \lambda = 0$$

$$X(x) = c_1 e^{\mu x} + c_2 e^{-\mu x} \qquad \lambda = \mu^2$$

$$X(x) = c_1 \cos \mu x + c_2 \sin \mu x \qquad \lambda = -\mu^2$$

Very interesting to go through cases to look for <u>non-zero</u> soln's satisfying B.C.

<u>General Sturm-Liouville Prob.</u>

<u>ODE</u>: $[p(x)y']' + (q(x) + \lambda r(x))y = 0$

$$\text{B.C.} \begin{cases} k_1 y(a) + k_2 y'(a) = 0 & k_1, k_2 \text{ not both } 0 \\ \ell_1 y(b) + \ell_2 y'(b) = 0 & \ell_1, \ell_2 \text{ not both } 0 \end{cases}$$

p, p', q, r continuous on $[a,b]$
and $r(x) > 0$ on (a,b).

Facts: e-vals λ must be real.

e-fcns are $\perp$ on (a,b) with respect to the weight function $r(x)$:

$$\int_a^b y_1(x)\, y_2(x) \quad r(x)\, dx$$

weight fcn.

Remark: $X'' + (\lambda) X = 0 \qquad p=1,\ q=0,\ r\equiv 1.$

EX: p. 503: 13.

$$(*) \qquad y'' + 8y' + (16+\lambda)\, y = 0 \quad \begin{cases} y(0)=0 \\ y(\pi)=0 \end{cases}$$

$$[py']' + (q + \lambda r)\, y = 0$$

$$p y'' + p' y' + (q + \lambda r)\, y = 0 \leftarrow (*)?$$

Big Idea p·(*)

$$p y'' + 8p\, y' + (16p + \lambda p)\, y = 0$$

Aha! Need $p' = 8p$. Easy! $p = e^{8t}$

$$\underbrace{[e^{8t}}_{p} y']' + (\underbrace{16e^{8t}}_{q} + \lambda \underbrace{e^{8t}}_{r=e^{8t}}) y = 0$$

Great! Can apply Theorem and know e-fcns $\perp$ with respect to weight function.

<u>Warning:</u> When looking for e-vals and e-fcns, use the original ODE $(*)$.

<u>Legendre Eqn:</u> $(1-x^2)y'' - 2xy' + n(n+1)y = 0$

$$[(1-x^2)y']' + \underbrace{\lambda}_{\lambda = n(n+1)} y = 0$$

Get Legendre Polynomials $P_n(x)$.

Note: $p(x) = 1-x^2$, $q(x) \equiv 0$, $r(x) \equiv 1$.

So Legendre Poly's $\perp$ on $(-1,1)$.

$$\int_{-1}^{1} P_n(x) P_m(x) \cdot \underbrace{1}_{r(x)} \, dx = 0 \text{ if } n \neq m.$$

HWK: Show $P_n(\cos\theta)$ are orthog. with respect to weight function $\sin\theta$.

Hmmm. $\int P_n(\underbrace{\cos\theta}_{u})\, P_m(\underbrace{\cos\theta}_{u})\ \sin\theta\, d\theta$

$du = ?$