Lesson 29 11.6 Generalized Fourier Series

HWK 9: 26, 27, 28 due Wed.

Legendre Polys: p. 178 $(1-x^2)y'' - 2xy' + n(n+1)y = 0$

"Singular Sturm-Liouville" $y'' + Py' + Qy = 0$

(See Theorem 1 in 11.5)

$P = \frac{-2x}{1-x^2} \leftarrow$ singular at ± 1

$$P_n(x) = \sum_{m=0}^{M} (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!} x^{n-2m}$$

where $M = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ odd} \end{cases}$

$P_0(x) = 1$ $\qquad$ $P_1(x) = x$

$P_2(x) = \frac{1}{2}(3x^2-1)$ $\qquad$ $P_3(x) = \frac{1}{2}(5x^3-3x)$

$P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$ $\qquad$ $P_5(x) = \frac{1}{8}(63x^5-70x^3+15x)$

<u>Facts</u>: 1) $\text{Span}(P_0, P_1, \dots, P_n) = \text{Span}(1, x, x^2, \dots, x^n)$

$= \text{Polys of deg} \le n$

2) $P_{even}(x)$ is an even fcn

$P_{odd}(x)$ is odd

3) The P_n are $\perp$ on $(-1,1)$ $\quad$ $[r(x) \equiv 1]$

4) The L^2-norm of P_n on $(-1,1)$ is

$$\|P_n\| = \sqrt{(P_n, P_n)} = \sqrt{\int_{-1}^{1} P_n(x)^2\,dx} = \sqrt{\frac{2}{2n+1}}$$

5) The P_n form a complete orthogonal set on $(-1,1)$, meaning that every f in $L^2(-1,1)$ can be expanded:

$$f(x) = \sum_{n=0}^{\infty} c_n P_n$$

What are c_n's?

Mult. by P_m: $\quad f P_m = \sum_{n=0}^{\infty} c_n P_n P_m$

Integrate: $\quad \int_{-1}^{1} f P_m\,dx = \sum_{n=0}^{\infty} c_n \underbrace{\int_{-1}^{1} P_n P_m\,dx}$

$$= c_m \int_{-1}^{1} P_m^2\,dx$$

$$\delta_{nm} = \begin{cases} 1 & n=m \\ 0 & n\neq m \end{cases}$$

"Kronecker Delta"

$$= c_m \|P_m\|^2 = c_m \cdot \frac{2}{2m+1}$$

$$\boxed{c_m = \frac{2m+1}{2} \int_{-1}^{1} f(x) P_m(x)\,dx}$$

$\leftarrow$ can change m to an n now.

6) $0 \le \| f - \sum_0^N c_n P_n \|^2 = \|f\|^2 - \sum_0^N \frac{|c_n|^2}{\left(\frac{2}{2n+1}\right)}$

Bessel's Ineq. $\sum_{n=0}^{N} \frac{2n+1}{2} |c_n|^2 \le \|f\|^2$

Can let $N \to \infty$!

7) $0 = \| f - \sum_0^\infty c_n P_n \|^2 = \|f\|^2 - \sum \frac{|c_n|^2}{\left(\frac{2}{2n+1}\right)}$

Parseval's Identity: $\sum_0^\infty \frac{2n+1}{2} |c_n|^2 = \|f\|^2$

completeness!

<u>Hilbert Space</u>: $L^2(-1,1)$. The P_n form an orthogonal basis. $f = \sum_0^\infty c_n P_n$

like coord vectors.
c_n = "coord"

$L^2(-1,1) \sim \ell^2$

$f \sim (c_n)_{n=0}^\infty$

Use these facts to do p. 509: 1, 3, 5.

<u>EX</u>: Expand x^4 in Legendre Poly's.

First use Fact 1: $x^4 = c_0 P_0 + c_1 P_1 + c_2 P_2 + c_3 P_3 + c_4 P_4$

<u>Fact</u>: c's are uniquely determined by $\perp$.

Mult. by P_k and integrate.

$$\int_{-1}^{1} x^4 P_k(x)\,dx = c_k \underbrace{\int_{-1}^{1} P_k(x)^2\,dx}_{\frac{2}{2k+1}}$$

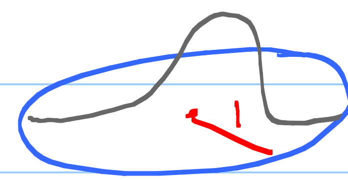
$$c_k = \frac{2k+1}{2}\int_{-1}^{1} x^4 P_k(x)\,dx$$

Simplifying fact

$$\int_{-1}^{1} \underbrace{\underset{\uparrow \text{ even}}{x^4}\;\underbrace{P_k(x)}_{\text{odd if } k \text{ odd}}}_{\text{odd}}\,dx = 0 \text{ if } k \text{ odd}$$

Remark: Another way to get L. Poly's: Use Gram-Schmidt (forever!) to orthogonalize
$1, x, x^2, x^3, x^4, \ldots$

Vibrating circular membrane:



$$u(r,\theta,t) = R(r)\,\Theta(\theta)\,T(t)$$

$$\underbrace{\Delta u}_{\text{polar coords}} = c^2 \frac{\partial^2 u}{\partial t^2}$$

Get 3 ODE problems

1) $\Theta'' + \lambda \Theta = 0$

$$\left.\begin{aligned} \Theta(0) &= \Theta(2\pi) \\ \Theta'(0) &= \Theta'(2\pi) \end{aligned}\right\} \text{periodic B.C. (see Thm 1)}$$

2) $r^2 R'' + r R' + (r^2 - \lambda) R = 0$ ← Bessel's Eqn.

$R(1) = 0$ ← B.C.

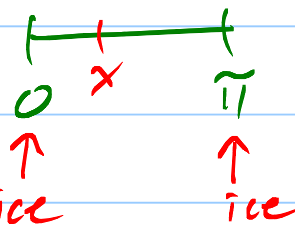
R bounded near origin ← B.C. of sorts
$r = 0$ is a singular spot of ODE.

Get Bessel fcns $J_n(r)$.

They satisfy orthog. cond.

3) $T'' + \lambda T = 0$ ← easy!

Heat prob: Hot wire



$\underbrace{u(x,0) = 1}_{\text{I.C.}}$

$$\left.\begin{aligned} u(0,t) &= 0 \\ u(\pi,t) &= 0 \end{aligned}\right\} \text{B.C.}$$

$$u(x,t) = X(x)\,T(t)$$

$X'' - \lambda X = 0$ ← $\lambda = -n^2$

$X(0) = 0,\ X(\pi) = 0$

$X_n(x) = \sin nx$

$T' - \lambda T = 0$

$T(t) = C e^{-\lambda t}$

(take $c = 1$).

$T_n(t) = e^{-n^2 t}$

$$u(x,t) = \sum_{n=1}^{\infty} c_n \sin(nx)\, e^{-n^2 t}$$

Satisfies PDE, B.C.

Last step: I.C. $\underbrace{u(x,0)}_{1} = \sum_{n=1}^{\infty} c_n \sin nx$

Get $c_n = \frac{2}{\pi}\int_0^{\pi} 1 \cdot \sin nx \, dx$

$$= \frac{2}{\pi}\left[-\frac{1}{n}\cos nx\right]_0^{\pi}$$

$$= \frac{2}{\pi n}\Big[1 - \underbrace{\cos n\pi}_{\substack{-1 \text{ n odd} \\ 1 \text{ n even}}}\Big]$$

$$= \begin{cases} \frac{4}{\pi n} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$$

$$u(x,t) = \frac{4}{\pi}\Big(e^{-t}\sin x + \frac{1}{3}e^{-3^2 t}\sin 3x + \frac{1}{5}e^{-5^2 t}\sin 5x + \cdots\Big)$$