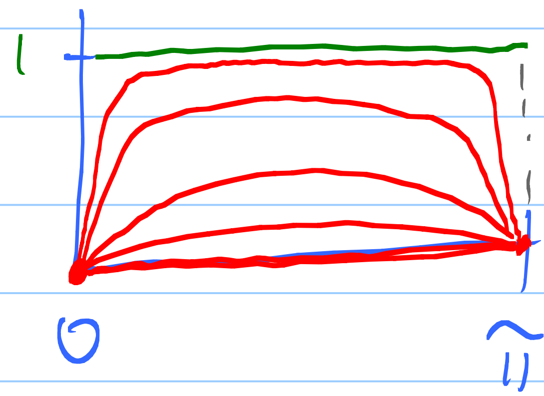
Lesson 30 11.7 Fourier Integral (29,30,31 wed. HWK 10)

$$u(x,t) = \frac{4}{\pi}\left(e^{-t}\sin x + \frac{1}{3}e^{-3^2 t}\sin 3x + \frac{1}{5}e^{-5^2 t}\sin 5t + \dots\right)$$



Bessel's Ineq: $\sum_{n=1}^{\infty} c_n^2 \leq \|f\|^2$

$\uparrow$ = Parseval's

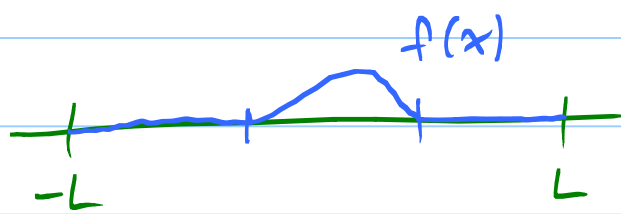
See $c_n \to 0$ as $n \to \infty$.

Fourier coeff $\to 0$ as $n \to \infty$.

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \underbrace{\sin nx}_{\text{not } \to 0} \, dx$$

Wiggling must cause cancelation.

Fourier Integral:

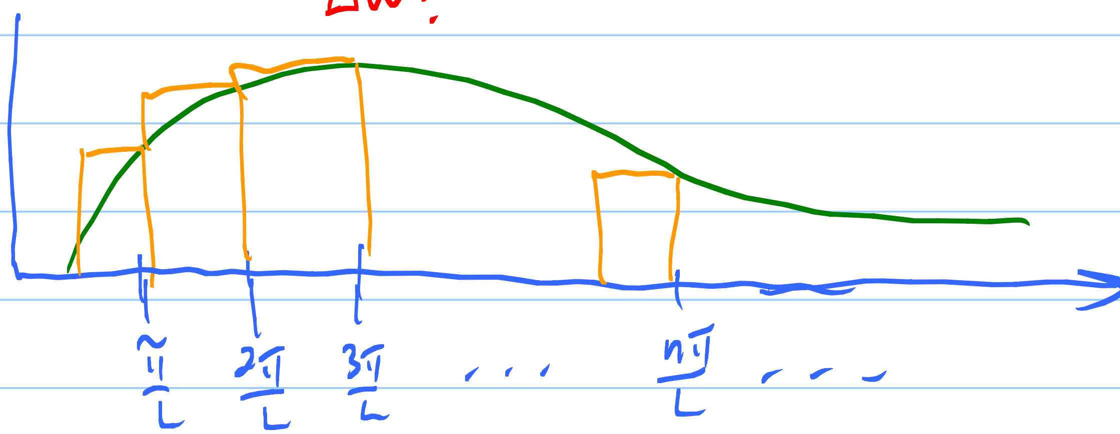


Let $L \to \infty$.

$$f(x) = \underbrace{\frac{1}{2L}\int_{-L}^{L} f(x)\,dx}_{\to 0 \text{ as } L\to\infty} + \sum_{n=1}^{\infty}\left[\left(\frac{1}{L}\int_{-L \to -\infty}^{L \to \infty} f(v)\cos\frac{n\pi v}{L}\,dv\right)\cos\frac{n\pi x}{L}\right]$$

w_n $\qquad$ w_n

$$+\left(\underbrace{\frac{1}{L}}_{\Delta w?}\int_{-L \,(\to -\infty)}^{L \,(\to \infty)} f(v)\sin\overbrace{\frac{n\pi}{L}}^{w_n} v\,dv\right)\sin\underbrace{\frac{n\pi}{L}}_{w_n} x\ \Bigg]$$



$$f(x) = \lim_{L\to\infty}\sum_{n=1}^{\infty}\left[\left(\underbrace{\frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos w_n v\,dv}_{A(w_n)}\right)\cos w_n x\,\underbrace{\left(\frac{\pi}{L}\right)}_{\Delta w} + \text{similar for sine part}\right]$$

Fourier Integral:

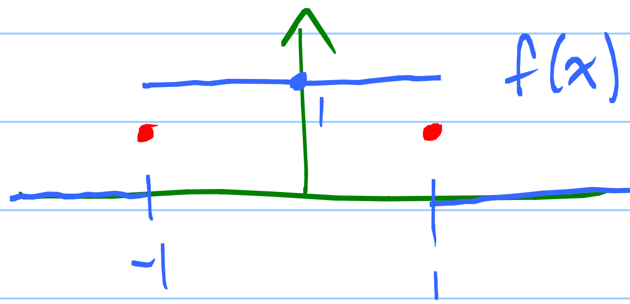
$$f(x) = \int_0^\infty A(w)\cos wx\,dw + B(w)\sin wx\,dw$$

where
$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\,dv$$

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv\,dv$$

Fact: If f is piecewise C^1-smooth and $\int_{-\infty}^{\infty}|f|\,dx$ is finite (i.e., f is absolutely integrable), then f is given by its F.I. (At jumps, the integral converges to the midpoint of the jump.

EX:



$$A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} \underbrace{\overset{\text{even}}{f(v)}\ \underbrace{\cos wv}_{\text{even}}}_{\text{even}}\, dv = \frac{2}{\pi}\int_{0}^{\infty} f(v)\cos wv\, dv$$

$$= \frac{2}{\pi}\int_0^1 1\cdot \cos wv\, dv = \frac{2}{\pi}\left[\frac{1}{w}\sin wv\right]_0^1$$

$$= \frac{2}{\pi w}\left[\sin w - \sin 0\right] = \underline{\frac{2\sin w}{\pi w}}$$

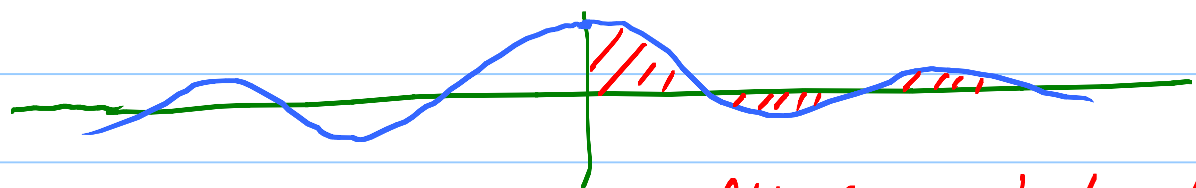
$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} \underbrace{\overset{\text{even}}{f(v)}\ \underbrace{\sin wv}_{\text{odd}}}_{\text{odd}}\, dv = 0$$

$$f(x) = \int_0^{\infty} \underbrace{\frac{2\sin w}{\pi w}}_{A(w)} \cos wx\, dw + \underset{B\text{ term}}{0}$$

$$= \begin{cases} 0 & x < -1 \\ 1/2 & x = -1 \\ 1 & -1 < x < 1 \\ 1/2 & x = 1 \\ 0 & x > 1 \end{cases}$$

Interesting spot: $x = 0$.

$$1 = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\sin w}{w}\, dw$$



Alt. Series test shows $\int$ exists.

Odd $f(x)$: $A(w) = 0$ $\quad \Big[\underbrace{\underset{\substack{\uparrow \\ \text{odd}}}{f} \cdot \underset{\substack{\uparrow \\ \text{even}}}{\cos wv}}_{\text{odd}}\Big]$

$$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} \underbrace{\underset{\substack{\uparrow \\ \text{odd}}}{f(v)}\ \underset{\substack{\uparrow \\ \text{odd}}}{\sin wv}}_{\text{even}}\, dv \;=\; \underbrace{\frac{2}{\pi}\int_{0}^{\infty} f(v) \sin wv\, dv}_{\text{Fourier Sine Transform}}$$

EVEN $f(x)$: $B(w) = 0$ $\quad \Big[\underset{\substack{\uparrow \\ \text{even}}}{f} \cdot \underset{\substack{\uparrow \\ \text{odd}}}{\sin wv}\Big]$ $\quad$ even $\cdot$ odd $=$ odd

$$\underbrace{A(w) = \frac{2}{\pi}\int_0^\infty f(v)\,\cos wv\,dv}_{\text{Fourier Cosine Transform}}$$

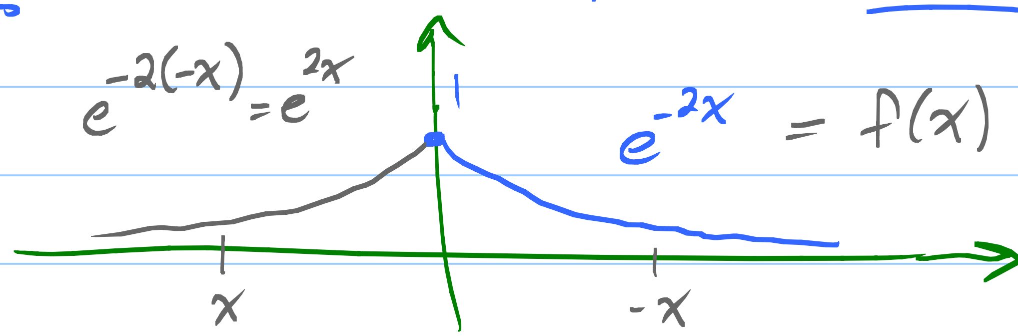
Prob: Expand $f(x) = e^{-2x}$ for $x > 0$ as a Fourier Cosine integral.

↑ even

↑ free to make f be anything for $x<0$.

Step 1: Extend f to $\mathbb{R}$ as an even fcn.

$f(-x) = f(x)$ $\quad e^{-2(-x)} = e^{2x}$ $\quad e^{-2x} = f(x)$



Step 3: Know $B(w) = 0$, ⟸ no Sine series part!

$$A(w) = \frac{2}{\pi}\int_0^\infty e^{-2v}\cos wv\,dv = \frac{2}{\pi}\cdot\frac{2}{2^2+w^2}$$

$s=2$ $\quad t=v$

$$\left[\ \mathcal{L}[\cos wt] = \int_0^\infty e^{-st}\cos wt\,dt = \frac{s}{s^2+w^2}\right.$$

$$e^{-2x} = \int_0^\infty \underbrace{\left(\frac{2}{\pi}\cdot\frac{2}{4+w^2}\right)}_{A(w)} \cos wx \, dw$$

valid for $x > 0$.

In practice, usually need to truncate the integral $\int_0^{\text{Big}}$

503:13. $\left[\begin{array}{l} y'' + 8y' + (\lambda+16)y = 0 \\ y(0)=0, \ y(\pi). \end{array}\right.$

$$r^2 + 8r + (\lambda+16) = 0$$

roots $r = -4 \pm \sqrt{-\lambda}$

<u>Case 1</u>: $\lambda = 0$. $y = c_1 e^{-4x} + c_2 x e^{-4x}$

Need $y(0)=0$, $y(\pi)=0$. Get $c_1=0$, $c_2=0$.

No eigenfunction! $\lambda=0$ is not an e-val.

<u>Case 2</u>: $\lambda < 0$. $\lambda = -k^2$. $r = -4 \pm k$

$$y = c_1 e^{(-4+k)x} + c_2 e^{(-4-k)x}$$

Only zero sol^n here too!

<u>Case 3</u>: $\lambda > 0$, $\lambda = k^2$. $r = -4 \pm ki$

$$y = c_1 e^{-4x} \cos kx + c_2 e^{-4x} \sin kx$$

Find special values of k that allow non-zero sol^{n}s to ODE with $y(0)=0$, $y(\pi)=0$.