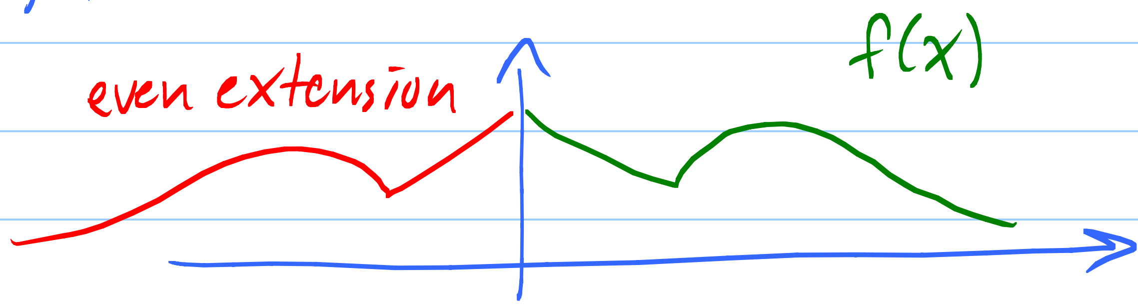
Lesson 31 11.8 Fourier Cosine, Sine Transforms

(29, 30, 31 due Wed)

EX:



$$f(x) = \int_0^\infty A(w) \cos wx \, dw + \left(\begin{array}{l} B \text{ part} = 0 \\ \text{if } f \text{ even} \end{array} \right)$$

$$A(w) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(v) \cos vw \, dv = \frac{2}{\pi} \int_0^\infty f(v) \cos vw \, dv$$

Fourier Cosine Transform:

$$\tilde{\mathcal{F}}_c[f] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(v) \cos wv \, dv$$

fcn of x for $x > 0$

fcn of w for $w > 0$.

Amazing Fact: $\tilde{\mathcal{F}}_c\left[\tilde{\mathcal{F}}_c[f]\right] = f$

Think: $\tilde{\mathcal{F}}_c$ is it's own inverse!

Fourier Sine Transform:

$$\tilde{\mathcal{F}}_s[f] = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin wx \, dx$$

Notations: $\tilde{\mathcal{F}}_c[f] = \hat{f}_c$

$\tilde{\mathcal{F}}_s[f] = \hat{f}_s$

Same Amazing Fact: $\tilde{\mathcal{F}}_s[\tilde{\mathcal{F}}_s[f]] = f$.

Properties:

1) $\tilde{\mathcal{F}}_c[f'] = w\,\tilde{\mathcal{F}}_s[f] - \sqrt{\frac{2}{\pi}}\, f(0)$

2) $\tilde{\mathcal{F}}_s[f'] = -w\,\tilde{\mathcal{F}}_c[f]$

Note: In 1,2 need f cont and absolutely integrable, f' piecewise cont., and $f(x) \to 0$ as $x \to \infty$.

Check 1: $\tilde{\mathcal{F}}_c[f'](w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f'(x) \cos wx \, dx$

$$= \sqrt{\frac{2}{\pi}} \lim_{B \to \infty} \int_0^B \underbrace{\cos wx}_{u} \; \underbrace{f'(x)\,dx}_{dv}$$

$du = -w \sin wx \, dx \qquad v = f(x)$

$$= \sqrt{\frac{2}{\pi}} \lim_{B \to \infty} \left[\underbrace{(\cos wx)}_{u} \underbrace{f(x)}_{v} \Big|_0^B - \int_0^B \underbrace{f(x)}_{v} \underbrace{[-w \sin wx \, dx]}_{du} \right]$$

$$= \sqrt{\frac{2}{\pi}} \lim_{B\to\infty} \left[\underbrace{(\cos wB) f(B)}_{\to 0 \text{ as } B\to\infty} - (\cos 0) f(0) + w\int_0^B f(x)\sin wx\,dx \right]$$

$$= -\sqrt{\frac{2}{\pi}}\, f(0) + w\, \tilde{\mathcal{F}}_s[f]$$

Similarly for (2).

Consequence:

a) $\tilde{\mathcal{F}}_c[f''] = -w^2\, \tilde{\mathcal{F}}_c[f] - \sqrt{\frac{2}{\pi}}\, f'(0)$

b) $\tilde{\mathcal{F}}_s[f''] = -w^2\, \tilde{\mathcal{F}}_s[f] + \sqrt{\frac{2}{\pi}}\, w f(0)$

Why a): $f'' = F'$ where $F = f'$.

$$\tilde{\mathcal{F}}_c[f''] = \tilde{\mathcal{F}}_c[F'] = w\, \tilde{\mathcal{F}}_s[\underset{\uparrow \atop f'}{F}] - \sqrt{\frac{2}{\pi}}\, \underbrace{F(0)}_{f'(0)}$$

$$= w\left(\underline{-w\, \tilde{\mathcal{F}}_c[f]}\right) - \sqrt{\frac{2}{\pi}}\, f'(0)$$

Similarly for (b).

Prob: Find $\tilde{\mathcal{F}}_c[\underbrace{e^{-ax}}]$ $\qquad f''(x) = a^2 f(x)$

$f(x) = e^{-ax}$ $\quad f'(x) = -a e^{-ax}$

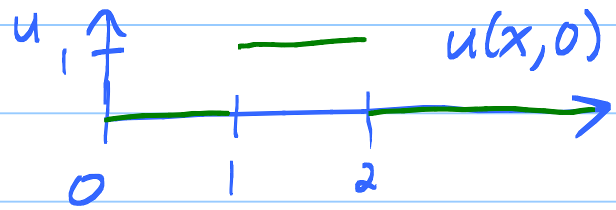
Aha! $\tilde{\mathcal{F}}_c[\underbrace{f''}_{a^2 e^{-ax}}] = -w^2\, \tilde{\mathcal{F}}_c[\underset{e^{-ax}}{f}] - \sqrt{\frac{2}{\pi}}\, f'(0)$

$$a^2 \, \tilde{\mathcal{F}}_c\left[e^{-ax}\right] = -w^2 \, \tilde{\mathcal{F}}_c\left[e^{-ax}\right] - \sqrt{\tfrac{2}{\pi}}\left(-a e^0\right)$$

$$\left(w^2 + a^2\right) \tilde{\mathcal{F}}_c\left[e^{-ax}\right] = \sqrt{\tfrac{2}{\pi}} \cdot a$$

$$\tilde{\mathcal{F}}_c\left[e^{-ax}\right] = \sqrt{\tfrac{2}{\pi}} \; \frac{a}{a^2 + w^2}$$

<u>Application:</u> Hot wire



Heat Eqn: $u_{xx} = u_t$

B.C. Ice the origin
$u(0,t) = 0$ for all t.

Apply $\tilde{\mathcal{F}}_s$ in space variable x.

$$\hat{u}(w,t) = \sqrt{\tfrac{2}{\pi}} \int_0^\infty u(x,t) \sin wx \, dx$$

<u>Note:</u> Can take $\frac{\partial}{\partial t}$ under $\int$ to see

$$\frac{\partial \hat{u}}{\partial t} = \tilde{\mathcal{F}}_s\left[\frac{\partial u}{\partial t}(x,t)\right]$$

But,

$$\tilde{\mathcal{F}}_s\left[\frac{\partial^2 u}{\partial x^2}\right] = -w^2 \, \tilde{\mathcal{F}}_s[u] + \sqrt{\tfrac{2}{\pi}}\, w \, \underbrace{u(0,t)}_{=0}$$

$\tilde{\mathcal{F}}_s$ applied to $u_{xx} = u_t$ spits out

$$\boxed{-w^2 \hat{u} = \frac{\partial \hat{u}}{\partial t}}$$

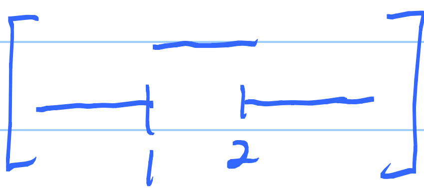
exponential decay prob' int.

$$\hat{u}(w,t) = C(w) e^{-w^2 t}$$

arbitrary const might depend on w!

Let $t=0$ to determine $C(w)$:

$$\hat{u}(w,0) = C(w)\, e^0 = C(w)$$

$\tilde{\mathcal{F}}_s$  $= C(w)$

$$= \sqrt{\frac{2}{\pi}} \int_1^2 1 \cdot \sin wx \, dx = \sqrt{\frac{2}{\pi}} \left(\frac{\cos w - \cos 2w}{w} \right)$$

(perfectly fine at $w=0$)

$$\hat{u}(w,t) = \sqrt{\frac{2}{\pi}} \left(\frac{\cos w - \cos 2w}{w} \right) e^{-w^2 t}$$

Finally, undo $\tilde{\mathcal{F}}_s$ by applying $\tilde{\mathcal{F}}_s$ again!

$$u(x,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sqrt{\frac{2}{\pi}} \left(\frac{\cos w - \cos 2w}{w} \right) e^{-w^2 t} \sin wx \, dw$$

The Fourier Transform. 11.9

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos nx + b_n \sin nx \right)$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$$

$$c_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) e^{-in\theta}$$