Lesson 32 11.9 Complex Fourier Transform

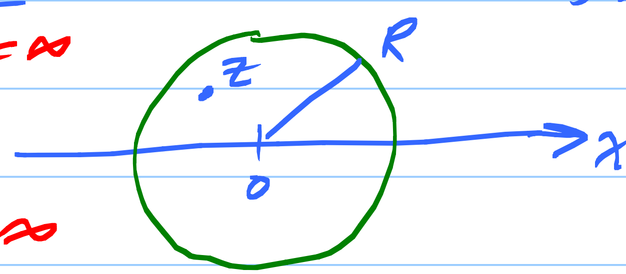
(29, 30, 31 due Wed.) 32 do but not due

Going complex: $ax^2 + bx + c = 0$ ← always has 2 roots over $\mathbb{C}$, can factor

Fundamental Theorem of Algebra:
$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$
has n complex roots and poly's factor over $\mathbb{C}$.

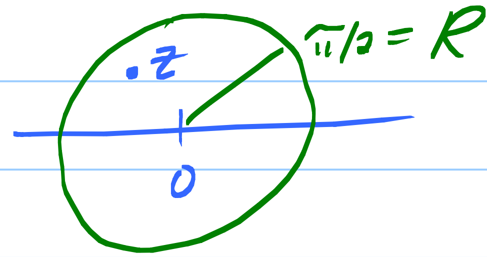
Good reasons to do Calculus over $\mathbb{C}$ (MA 528).

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \quad \leftarrow R = \infty$$



$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \quad \leftarrow R = \infty$$

$$\tan x = \frac{\sin x}{\cos x} = x + \cdots$$



$$\mathcal{L}[f] = \int_0^\infty e^{-st} f(t)\,dt$$

let $s = z$, a complex number

Fourier Series:

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx)$$

$$= \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$c_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(x) e^{-inx}\,dx$$

Fact: e^{inx} are orthogonal on $[-\pi,\pi]$.

$$\int_{-\pi}^{\pi} e^{inx}\,\overline{e^{imx}}\,dx = \int_{-\pi}^{\pi} e^{inx}\,e^{-imx}\,dx$$

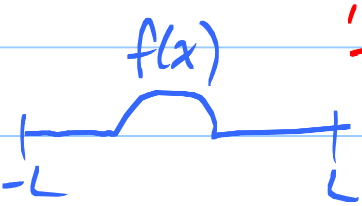
conjugate of e^{imx}

$$= \int_{-\pi}^{\pi} e^{i(n-m)x}\,dx$$

$$= \left[\frac{1}{i(n-m)}\,e^{i(n-m)x}\right]_{-\pi}^{\pi}$$

0 because everything is 2π periodic if $n \neq m$.

Get complex complex Fourier Series on $[-L,L]$. Take $f(x)$



and let $L \to \infty$.

Get the Fourier Transform:

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{-iwx}\,dx$$

The Fourier Transf of f.

Notation: $\mathcal{F}[f] = \hat{f}$

$$\mathcal{F}\Big[\underbrace{\text{function of } x \text{ on } (-\infty,\infty)}_{\text{Need } \int_{-\infty}^{\infty} |f|\,dx < \infty.}\Big] = \text{function of } w \text{ on } (-\infty,\infty)$$

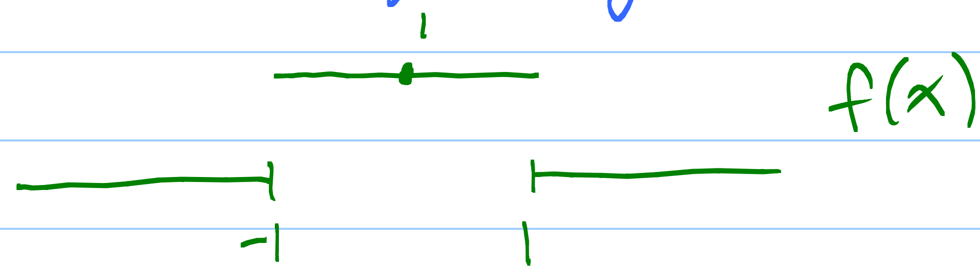
Big Fact: $$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{ixw} \, dw$$

Inverse Fourier Transform:

$$\mathcal{F}^{-1}[g] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(w) e^{ixw} \, dw$$

Notation: $\mathcal{F}^{-1}[g] = \check{g}$.

EX:



$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ixw} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} 1 \cdot e^{-ixw} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{-1}{iw} e^{-ixw} \right]_{x=-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{-1}{iw} \left(e^{-iw} - e^{iw} \right)$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{w} \cdot 2 \, \frac{e^{iw} - e^{-iw}}{2i} = \sqrt{\frac{2}{\pi}} \, \frac{\sin w}{w}$$

Fact: $\|f\|^2 = \|\hat{f}\|^2$ Plancherel Identity.

$$\int_{-\infty}^{\infty} |f|^2\,dx = \int_{-\infty}^{\infty} |\hat{f}|^2\,dw$$

Fourier Transform is L^2 Isometry

$$\int_{-1}^{1} 1^2\,dx = \int_{-\infty}^{\infty} \left(\sqrt{\frac{2}{\pi}}\,\frac{\sin w}{w}\right)^2 dw$$

$$2 = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{\sin^2 w}{w^2}\,dx$$

$$\int_{-\infty}^{\infty} \frac{\sin^2 w}{w^2}\,dw = \pi \qquad !$$

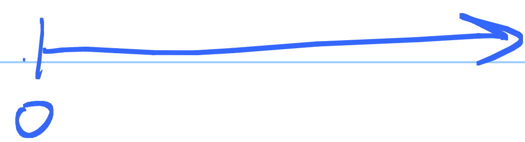
Properties: $\mathcal{F}$ is linear. Easy.

$$\mathcal{F}[f'] = iw\,\mathcal{F}[f]$$

$$\mathcal{F}[f''] = -w^2\,\mathcal{F}[f]$$

$i^2 = -1$

Hot wire:



$u_{xx} = u_t$

Iced end: $u(0,t) = 0$ for all t.

Fourier Sine Transf:

$$\mathcal{F}_s[u_{xx}] = -w^2 \mathcal{F}_s[u] + \sqrt{\frac{2}{\pi}}\, w \underbrace{u(0,t)}_{0}$$

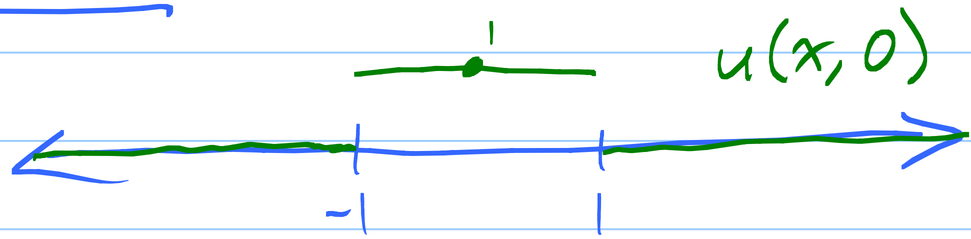
<u>Insulated end</u>: $\underbrace{\frac{\partial u}{\partial x}(0,t)}_{\uparrow \text{ temp gradient} = 0} = 0$ for all t

No heat flow.

Fourier Cosine Transform:

$$\mathcal{F}_c[u_{xx}] = -w^2 \mathcal{F}_c[u] - \sqrt{\frac{2}{\pi}}\, \underbrace{u_x(0,t)}_{0}$$

<u>No end</u>: Use <u>the</u> Fourier Transf.



I.C.

$$u_{xx} = u_t$$

$\leftarrow$ Hit with $\mathcal{F}$ in the space variable.

$$\hat{u}(w,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t)\, e^{-ixw}\, dx$$

$$\underbrace{\mathcal{F}[u_{xx}]}_{-w^2 \mathcal{F}[u]} = \mathcal{F}\left[\frac{\partial u}{\partial t}\right] = \frac{\partial}{\partial t} \mathcal{F}[u]$$

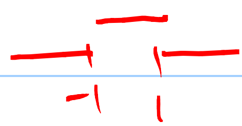
$$-w^2\,\hat{u}(w,t) = \frac{\partial}{\partial t}\hat{u}(w,t)$$ ← exponential decay in t variable!

$$\hat{u}(w,t) = C(w)\,e^{-w^2 t}$$

What is $C(w)$? Let $t=0$:

$$\hat{u}(w,0) = C(w)$$

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,0)\,e^{-ixw}\,dx = \underbrace{\sqrt{\frac{2}{\pi}}\;\frac{\sin w}{w}}_{C(w)}$$



To get u from $\hat{u}$, undo $\mathcal{F}$ with $\mathcal{F}^{-1}$:

$$u(x,t) = \mathcal{F}^{-1}\left[\sqrt{\frac{2}{\pi}}\;\frac{\sin w}{w}\;e^{-w^2 t}\right]$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sqrt{\frac{2}{\pi}}\;\frac{\sin w}{w}\;e^{-w^2 t}\;e^{iwx}\,dw$$

Convolution: Go back and call

$$(f*g)(t) = \int_0^t f(\tau)\,g(t-\tau)\,d\tau$$ Laplace Convolution

Fact: $\mathcal{L}[f*g] = F(s)\,G(s)$

Fourier Convolution: $(f*g)(x) =$

$$= \int_{-\infty}^{\infty} f(\tau)\, g(x-\tau)\, d\tau$$

Fact: $\mathfrak{F}[f*g] = \sqrt{2\pi}\,\hat{f}(w)\,\hat{g}(w)$

Note: Skip Discrete and Fast Fourier Transf
in 11.9