Lesson 33 Review 1 Exam 2 in class Monday

(32 not to be turned in)

Fourier convolution: $(f*g)(x) = \int_{-\infty}^{\infty} f(v)g(x-v)\,dv$

$$= \int_{-\infty}^{\infty} g(v)f(x-v)\,dv$$

$$\mathcal{F}[f*g] = \sqrt{2\pi}\,\hat{f}\cdot\hat{g}$$

Laplace Transform: $\dfrac{\alpha s+\beta}{as^2+bs+c}$

$$\frac{\alpha s+\beta}{(s-a)(s-b)} = \frac{A}{s-a} + \frac{B}{s-b}$$ ← Partial Fractions

$$\frac{\alpha s+\beta}{(s-a)^2} = \frac{A}{(s-a)^2} + \frac{B}{(s-a)}$$

$$\frac{\alpha s+\beta}{\text{complex roots}} = \frac{\alpha s+\beta}{\text{complete square}}$$

s-shift: $\mathcal{L}[e^{at}f(t)] = F(s-a)$ shift

t-shift: $\mathcal{L}[u(t-a)f(t-a)] = e^{-as}F(s)$

EX: s-shift: $\mathcal{L}[\sin bt] = \dfrac{b}{s^2+b^2}$

$$\mathcal{L}[e^{-at}\sin bt] = \frac{b}{(s+a)^2+b^2}$$

EX: $\mathcal{L}[\underbrace{t e^t}_{f(t-1)} u(t-1)] = e^{-1 \cdot s} F(s)$

Trick: $t = (t-a) + a$

$$te^t = [(t-1)+1] e^{[(t-1)+1]}$$

$$= [(t-1)+1] e \, e^{(t-1)} \leftarrow f(t-1)$$

So $f(t) = [(t)+1] e \cdot e^{(t)}$.

$$= e t e^t + e e^t$$

and $F(s) = e \cdot \frac{1}{(s-1)^2} + e \frac{1}{(s-1)}$

Ans: $e^{-s} F(s)$.

Note: $\mathcal{L}[te^t] = \mathcal{L}[e^t t] = \frac{1}{(s-1)^2}$

$\uparrow$ $\mathcal{L}[t] = \frac{1}{s^2}$

or $\mathcal{L}[t \underbrace{f(t)}_{e^t}] = -F'(s) = -\frac{d}{ds}\left[\frac{1}{(s-1)}\right]$ ✓

Convolution: $y'' + 2y' + 5y = r(t) \begin{cases} y(0) = 0 \\ y'(0) = 0 \end{cases}$

$$Y = \underbrace{\frac{1}{s^2+2s+5}}_{H(s)} R(s)$$

$$y = (h * r)(t) = \int_0^t h(\tau)\, r(t-\tau)\, d\tau$$

Laplace Conv.

$$\overset{\text{or}}{=} \int_0^t r(\tau)\, h(t-\tau)\, d\tau$$

Find $h(t)$:

$$H(s) = \frac{1}{s^2+2s+5} = \frac{1}{(s+1)^2+4}$$

$$\tfrac{1}{2}\cdot 2$$

$$= \frac{1}{2} \cdot \frac{2}{(s+1)^2+2^2} \qquad \mathcal{L}[\sin bt] = \frac{b}{s^2+b^2}$$

$$= \mathcal{L}\left[\tfrac{1}{2} e^{-1\cdot t} \sin 2t\right]. \quad \text{So} \quad h(t) = \tfrac{1}{2} e^{-t} \sin 2t$$

$$\mathcal{L}\Big[\underbrace{\int_0^t f(\tau)\, d\tau}_{f * g \text{ where } g \equiv 1}\Big] = \frac{F(s)}{s} = \underset{\uparrow\ \mathcal{L}[1]}{\frac{1}{s}} \cdot F(s)$$

Parseval's Identity: $2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)^2\, dx$

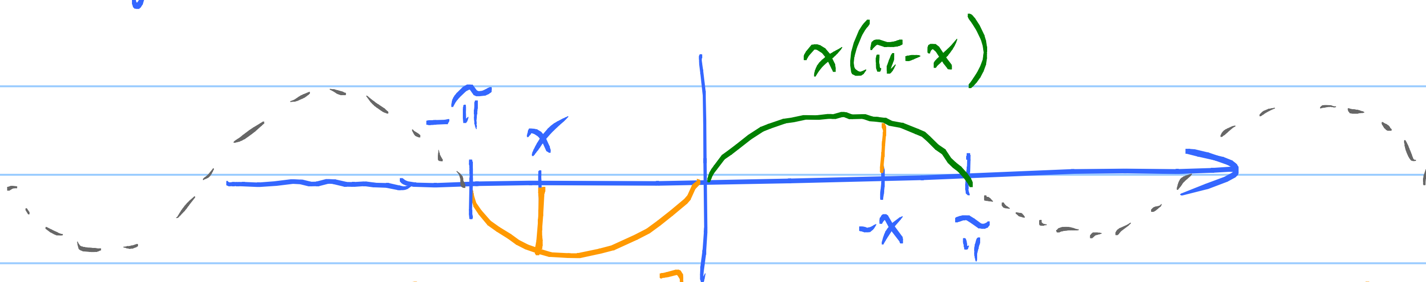
Prob: $x(\pi - x) = \frac{4}{\pi}\left(\frac{1}{1^3}\sin x + \frac{1}{3^3}\sin 3x + \frac{1}{5^3}\sin 5x + \cdots\right)$

on $[0, \pi]$. Fourier Sine Series.

What is $\left(\frac{1}{1^6} + \frac{1}{3^6} + \frac{1}{5^6} + \cdots\right)$?

Need a <u>full</u> Fourier Series for Parseval's.

Big Idea. Take <u>odd</u> extension of $f(x) = x(\pi - x)$.



$f(x) = -\left[(-x)(\pi - (-x))\right]$ ← formula for odd extension, $x < 0$.

Now $2 \cdot 0^2 + \left(0^2 + \left(\frac{1}{1^3}\right)^2 + \left(\frac{1}{5^3}\right)^2 + \cdots\right) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)^2\, dx$

all $\sum a_n^2$

$= \frac{2}{\pi}\int_0^{\pi} f(x)^2\, dx$

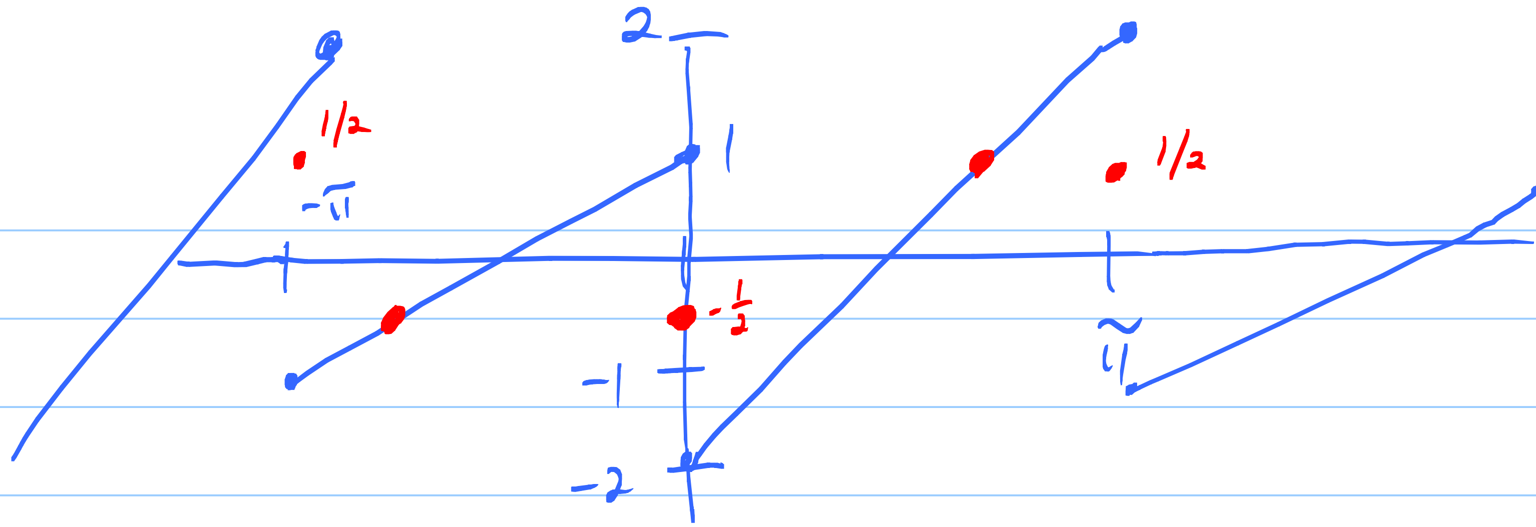
f odd: $f(-x) = -f(x)$

f^2: $f(-x)^2 = f(x)^2$ ← even

$= \frac{2}{\pi}\int_0^{\pi} x^2(\pi - x)^2\, dx$

Sum $= \frac{\pi^6}{960}$!

<u>Convergence:</u>



Step 1: Compute Fourier Series.

Step 2: What does it converge to?

Midpoint of jumps. (After adding 2π periodic extension.)

Sturm Liouville Prob: $y'' + \lambda y = 0 \quad \begin{cases} y'(0) = 0 \\ y'(2) = 0 \end{cases}$

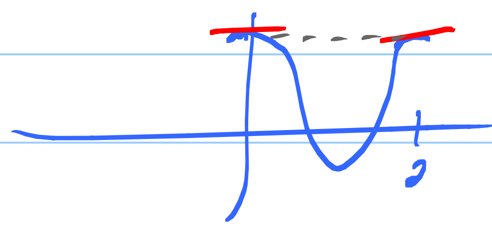
$r^2 + \lambda = 0$

Case $\lambda = 0$: $y = c_1 x + c_2 \qquad y'(x) = c_1$

$y'(0) = c_1 \overset{\text{want}}{=} 0 \qquad$ Must have $\underline{c_1 = 0}$.

$y'(2) = c_1 = 0$ too.

Aha! $y \equiv c_2$



is a sol^n even if $c_2 \neq 0$.

Get non-zero sol^n $\quad y \equiv 1$

[$\lambda = 0$ is e-val and $y_0 = 1$ is e-fcn for $\lambda = 0$.

<u>Case $\lambda > 0$:</u> $\lambda = \mu^2$. $r^2 + \mu^2 = 0$

$r = \pm \mu i$

$y = c_1 \cos \mu x + c_2 \sin \mu x$

$y' = -c_1 \mu \sin \mu x + c_2 \mu \cos \mu x$

$y'(0) = 0 + c_2 \mu \cos 0 = c_2 \mu = 0$ ← want

$\boxed{c_2 = 0}$

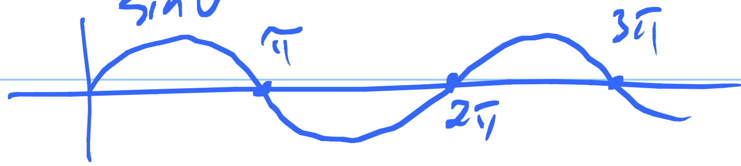
$y'(2) = -c_1 \mu \sin 2\mu = 0$ ← want

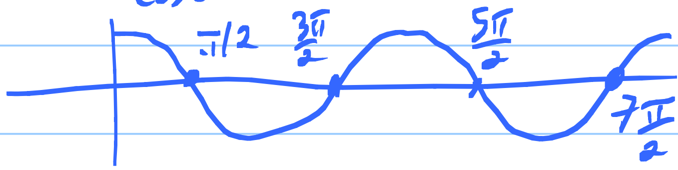
don't want $c_1 = 0$

need $\sin 2\mu = 0$

Need $2\mu = n\pi$

$\boxed{\mu = \frac{n\pi}{2}}$





zeroes of $\cos\theta$ $\left(\frac{\pi}{2} + n\pi\right)$ $n = 0, 1, 2, \ldots$

e-val $\lambda = \mu^2 = \left(\frac{n\pi}{2}\right)^2$

$n = 1, 2, 3, \ldots$

e-fcn for $\lambda = \left(\frac{n\pi}{2}\right)^2$: $y_n = \cos\left(\frac{n\pi}{2}\right)x$

take $c_1 = 1$.
(anything but 0.)

<u>Case $\lambda < 0$:</u> See no non-zero sol'ns.

Got $1, \cos\frac{n\pi}{2}x \quad n=1,2,3,\ldots$

These are $\perp$ on $[0,2]$. Cosine Series!

Remark: No weight function here.

$$[py']' + (q + \lambda r)y = 0$$

weight fcn.

$$y'' + \lambda y = 0$$

$r=1,\ q=0,\ p=1.$